Unit-5

Multirate Digital Signal Processing

Multirate DSP: The process of signal at different sampling rates in different parts of a system is called multirate digital signal processing.

Topics:

→ Decimation

→ Interpolation

→ Sampling rate conversion

→ Implementation of sampling rate conversion.

Sampling rate conversion:

The process of converting one sampling rate to another sampling is called sampling rate conversion

There are two types of sampling rate conversion:

1. Decimation (down sampling)
2. Interpolation (UP sampling)

1. Decimation:

Decimation is the process of decreasing sampling rate by a factor 'D'.

$$x(n) \longrightarrow \boxed{\downarrow D} \longrightarrow y(n) = x(nD)$$

Fig: Down sampler

Ex:-

Determine decimation of following sequence by a factor of 2 and 3.

$x(n) = \{1, 4, \cancel{7}, \cancel{3}\, 5, 7, 3, 2, 4, 5\}$

Sol: Case-1:- [For decimation factor D=2]

$x(n) = \{\underset{\uparrow}{1}, 4, 5, 7, 3, 2, 4, 5\}$

$x(n) \longrightarrow \boxed{\downarrow 2} \longrightarrow y(n) = x(2n)$

$\boxed{y(n) = x(2n)}$

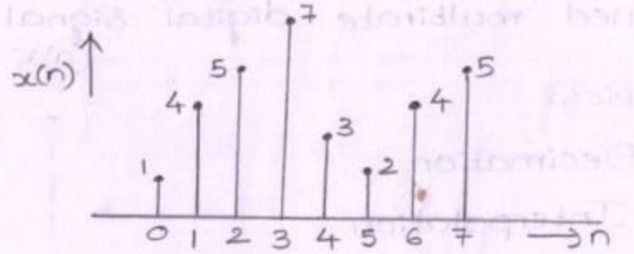


Fig: Plot of x(n)

If $n=0 \Rightarrow y(0) = x(0) = 1$

If $n=1 \Rightarrow y(1) = x(2) = 5$

If $n=2 \Rightarrow y(2) = x(4) = 3$

If $n=3 \Rightarrow y(3) = x(6) = 4$

If $n=4 \Rightarrow y(4) = x(8) = 0$

$\therefore y(n) = \{\underset{\uparrow}{1}, 5, 3, 4\}$

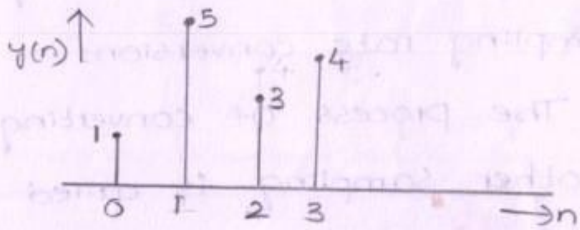


Fig: Plot of y(n)

Therefore, down sampler sequence can be obtained by selecting every D^{th} sample of i/p sequence.

case-2:- [For decimation factor D=3]

$x(n) = \{\underset{\uparrow}{1}, 4, 5, 7, 3, 2, 4, 5\}$

$x(n) \longrightarrow \boxed{\downarrow 3} \longrightarrow y(n) = x(3n)$

$\boxed{y(n) = x(3n)}$

$\therefore y(n) = \{1, 7, 4\}$ [Select every 3^{rd} sample]

Interpolation (UP sampler):-

The process of increasing sampling rate by a factor of 'I'.

$$x(n) \longrightarrow \boxed{\uparrow I} \longrightarrow y(n) = x\left(\frac{n}{I}\right)$$

Fig:- <u>UP</u> <u>sampler</u>

<u>Ex:-</u>

Determine Interpolation sequence for the following sequence by a factor of 2 and 3.

$x(n) = \{4, 3, 5, 7\}$

<u>sol:-</u> <u>Case-1</u>:- [For Interpolation factor $I=2$]

Given,

$x(n) = \{\underset{\uparrow}{4}, 3, 5, 7\}$ $\Longrightarrow$

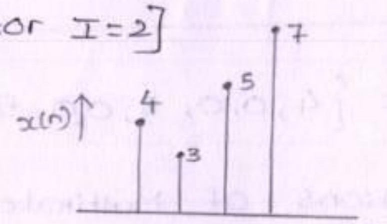


Fig:- <u>Plot</u> <u>of</u> <u>$x(n)$</u>

$$x(n) \longrightarrow \boxed{\uparrow 2} \longrightarrow y(n) = x\left(\frac{n}{2}\right)$$

$y(n) = x\left(\frac{n}{2}\right)$

If $n=0 \Rightarrow y(0) = x(0) = 4$

If $n=1 \Rightarrow y(1) = x(0.5) = 0$

If $n=2 \Rightarrow y(2) = x(1) = 3$

If $n=3 \Rightarrow y(3) = x(1.5) = 0$

If $n=4 \Rightarrow y(4) = x(2) = 5$

If $n=5 \Rightarrow y(5) = x(2.5) = 0$

If $n=6 \Rightarrow y(6) = x(3) = 7$

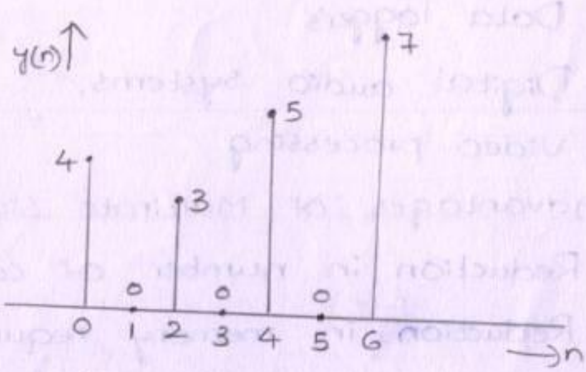


Fig:- <u>Plot</u> <u>of</u> <u>$y(n)$</u>

$\therefore y(n) = \{\underset{\uparrow}{4}, 0, 3, 0, 5, 0, 7\}$

Therefore, UP sampler sequence can be obtained by adding $(I-1)$ zeros in between the samples of i/p sequence.

Case-2:- [For Interpolation factor, $I=3$]

$$x(n) \longrightarrow \boxed{\uparrow 3} \longrightarrow y(n) = \frac{n}{3}$$

Here, $I=3$

$\therefore$ No. of zeros to be added in b/w the sequences $= I-1$
$= 3-1$
$= 2$

$\therefore y(n) = \{4, 0, 0, 3, 0, 0, 5, 0, 0, 7\}$

→ Applications of Multirate signal processing:

1. Sub-band coding in speech processing
2. Image compression
3. Narrow band FIR & IIR filters for various applications
4. Data logger's
5. Digital audio systems
6. Video processing

→ Advantages of Multirate signal processing:

1. Reduction in number of computations
2. Reduction in memory requirement
3. Reduction in order of the system
4. Reduction in finite word length effects

→ Expression for decimation process:

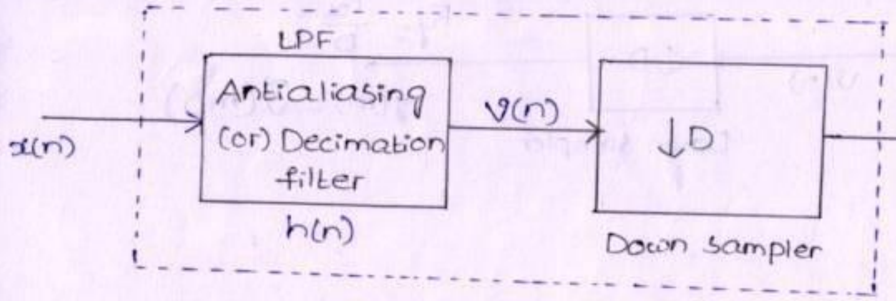


Fig: Decimation system

Decimation system consisting of low pass filter followed by down sampler.

Low pass filter band limits the i/P signal $x(n)$ and also avoids aliasing effect.

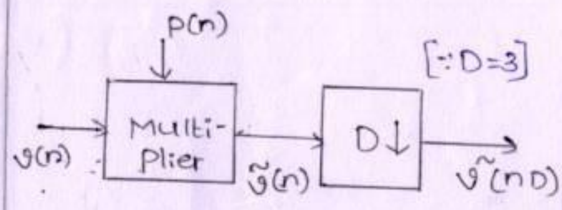
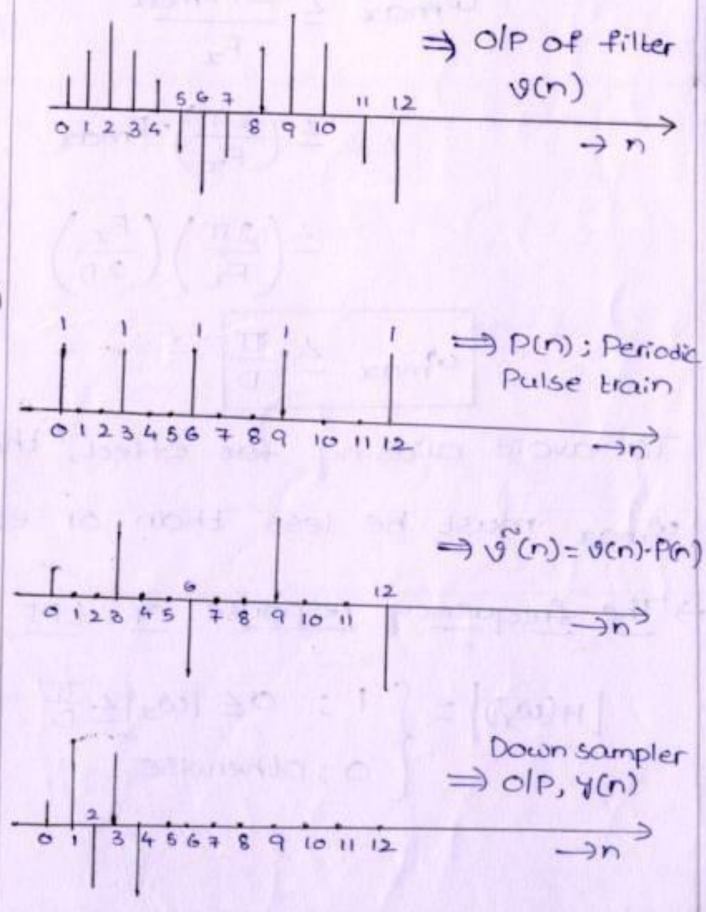
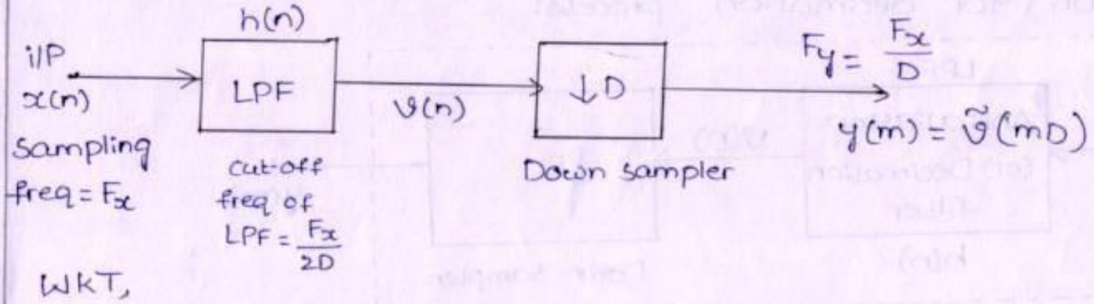


Fig: Internal operation of down sampler





WKT,

Sampling frequency = F_x

Sampling theorem is $F_x \geq 2F_{max}$

$$\implies F_{max} \leq \frac{F_x}{2}$$

After decimation process, $F_{max} \leq \frac{F_x}{2D}$

Relative frequency w.r.to F_x is

$$\omega_{max} \leq \frac{2\pi F_{max}}{F_x}$$

$$\leq \left(\frac{2\pi}{F_x}\right) \cdot F_{max}$$

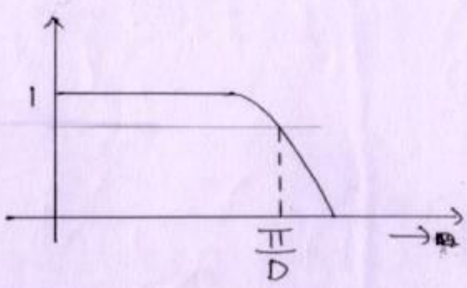
$$\leq \left(\frac{2\pi}{F_x}\right)\left(\frac{F_x}{2D}\right)$$

$$\boxed{\omega_{max} \leq \frac{\pi}{D}}$$

To avoid aliasing ~~far~~ effect, the cut-off frequency of LPF ω_{max} must be less than or equal to $\frac{\pi}{D}$ i.e., $\omega_{max} \leq \frac{\pi}{D}$

→ <u>The frequency response of LPF or anti-aliasing filter is:</u>

$$|H(\omega_x)| = \begin{cases} 1 ; & 0 \leq |\omega_x| \leq \frac{\pi}{D} \\ 0 ; & \text{otherwise} \end{cases}$$



→ The response of LPF is:

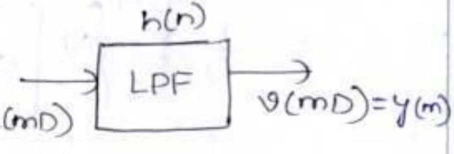
$$v(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k) \quad \text{—— ①}$$

→ Output of down sampler is:

$$y(m) = \tilde{v}(mD) = v(mD)\, P(mD) = v(mD)$$

$$y(m) = v(mD) = x(mD) * h(n)$$



$$\boxed{y(m) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(mD-k)}$$

The above expression represents time domain equation for down sampler.

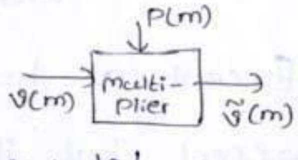
→ <u>For frequency domain</u>:-

$$Z\{y(m)\} = Y(z) = \sum_{m=-\infty}^{\infty} y(m) \cdot z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} \tilde{v}(mD) \cdot z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} \tilde{v}(m) \cdot z^{-m/D} \quad [\because \text{from time scaling property}].$$

$$= \sum_{m=-\infty}^{\infty} \tilde{v}(m) \cdot P(m) \cdot z^{-m/D} \text{ — ①}$$



P(m) is a periodic impulse train with period 'D'. So, the DFS representation of P(m) is given by

$$P(m) = \frac{1}{D} \sum_{k=0}^{D-1} e^{\frac{j2\pi mk}{D}} \text{ — ②} \qquad \left[\because C_k = \tfrac{1}{D} \text{ (constant)}\right]$$

On substituting eq-② in eq-①, we have

$$Y(z) = \sum_{m=-\infty}^{\infty} v(m)\left[\frac{1}{D}\sum_{k=0}^{D-1} e^{\frac{j2\pi mk}{D}}\right] z^{-m/D}$$

$$\Rightarrow \; = \frac{1}{\cancel{D}}\cancel{\sum_{m=-\infty}} \frac{1}{D}\sum_{k=0}^{D-1}\left[\sum_{m=-\infty}^{\infty} v(m)\left[e^{\frac{-j2\pi k}{D}}\cdot z^{+1/D}\right]^{-m}\right]$$

$$\Rightarrow \; = \frac{1}{D}\sum_{k=0}^{D-1}\left\{V\left[\underbrace{e^{\frac{-j2\pi k}{D}}\cdot z^{1/D}}_{z}\right]\right\}$$

$\left[\because Y(z) = Z\{y(m)\}\right]$

$\left[\because Y(z) = \sum_{m=-\infty}^{\infty} y(m)\cdot z^{-m}\right]$

$$= \frac{1}{D}\sum_{k=0}^{D-1}\left\{X\left(e^{\frac{-j2\pi k}{D}}\cdot z^{1/D}\right)\cdot H\left(e^{\frac{-j2\pi k}{D}}\cdot z^{1/D}\right)\right\}$$

$\Big[\because$ $x(n) \rightarrow \boxed{\text{LPF}} \rightarrow y(n)$, with $h(n)$ into the LPF

$y(n) = x(n) * h(n)$

$Y(z) = X(z)\cdot H(z)\Big]$

Put $z = e^{j\omega}$

$$\Rightarrow Y(e^{j\omega}) = \frac{1}{D}\sum_{k=0}^{D-1}\left\{X\left(e^{\frac{-j2\pi k}{D}}\cdot e^{\frac{j\omega}{D}}\right)\cdot H\left(e^{\frac{-j2\pi k}{D}}\cdot \cancel{z}\, e^{\frac{j\omega}{D}}\right)\right\}$$

$$Y(e^{j\omega}) = \frac{1}{D}\sum_{k=0}^{D-1}\left\{X\left(e^{j\left(\frac{\omega-2\pi k}{D}\right)}\right)\cdot H\left(e^{j\left(\frac{\omega-2\pi k}{D}\right)}\right)\right\}$$

$$\Rightarrow Y(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\frac{\omega-2\pi k}{D}\right)\cdot H\left(\frac{\omega-2\pi k}{D}\right)$$

Except at $k=0$, remaining terms produces aliasing effect but it will be eliminated by the low pass filter.

$$\therefore \boxed{Y(\omega) = \frac{1}{D}X\left(\frac{\omega}{D}\right)\cdot H\left(\frac{\omega}{D}\right) = \frac{1}{D}X\left(\frac{\omega}{D}\right)}$$

Relative frequency 'ω_y' w.r.to F_y is:

$$\omega_y = \frac{2\pi F_{max}}{F_y}$$

//ly, $\omega_x = \frac{2\pi F_{max}}{F_x}$

But output frequency of down sampler is, $F_y = \frac{F_x}{D}$

$$F_y = \frac{F_x}{D}$$

$$\frac{2\pi F_{max}}{\omega_y} = \frac{2\pi F_{max}}{\omega_x D}$$

$$\Rightarrow \frac{1}{\omega_y} = \frac{1}{\omega_x D}$$

$$\Rightarrow \boxed{\omega_y = \omega_x D}$$

→Let the spectrum range of ~~i/p signal~~ x is: Filter is: $H(\omega) = 0 \le |\omega_x| \le \frac{\pi}{D}$

or

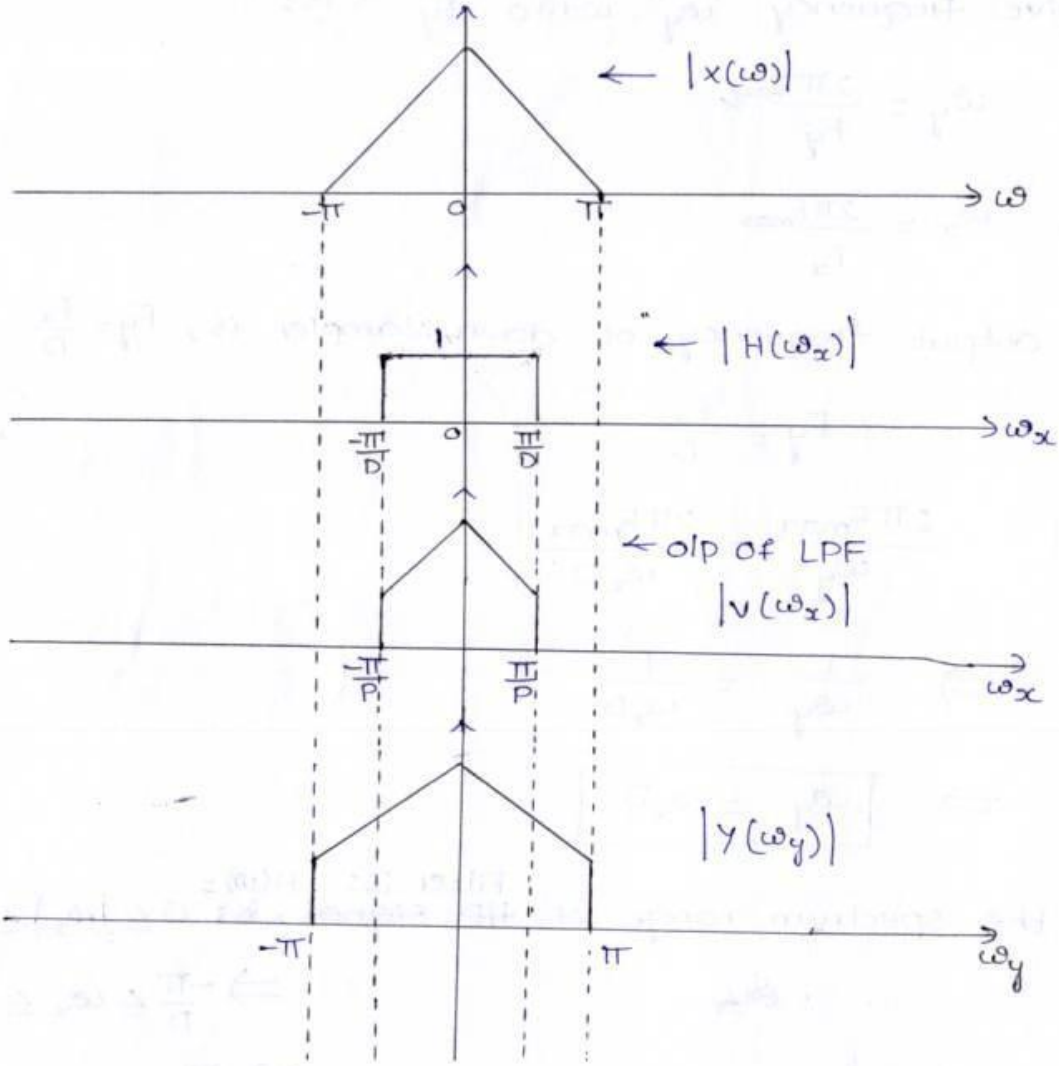
$$\Rightarrow -\frac{\pi}{D} \le \omega_x \le \frac{\pi}{D}$$

$$\Rightarrow -\frac{\pi}{D} \cdot D \le \omega_x D \le \frac{\pi}{D} \cdot D$$

$$\Rightarrow -\pi \le \omega_x D \le \pi$$

$$\Rightarrow \boxed{-\pi \le \omega_y \le \pi}$$

$[\because \omega_y = \omega_x D]$



→ Expression for Interpolation:

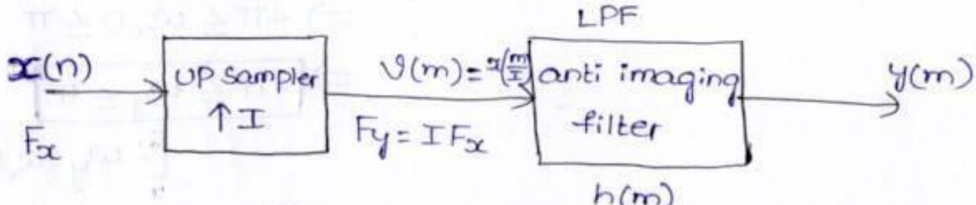


Fig: Interpolation system

Let $x(n)$ be the input signal with frequency range $-\pi$ to π.

Output of antiimaging filter,

$$y(m) = v(m) * h(m)$$

$$= \sum_{k=-\infty}^{\infty} v(k) \cdot h(m-k)$$

$$y(m) = \sum_{k=-\infty}^{\infty} x\left(\frac{k}{I}\right) \cdot h(m-k)$$

Replace k by kI

$$\boxed{y(m) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(m-kI)}$$

The above equation represents time domain expression for output of interpolation system.

The output of up sampler, $v(m) = \begin{cases} x\left(\frac{m}{I}\right) & ; m = 0, \pm I, \pm 2I, \pm 4I \ldots \\ 0 & ; \text{Otherwise} \end{cases}$

$$Z\{v(m)\} = V(Z) = \sum_{m=-\infty}^{\infty} v(m) \cdot z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} x\left(\frac{m}{I}\right) \cdot z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \cdot z^{-mI}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \cdot \left(z^{I}\right)^{-m}$$

$$V(Z) = X(Z^{I})$$

Let $n = \frac{m}{I}$

$m = nI$

If $m = -\infty, \Rightarrow n = -\infty$

$m = \infty, \Rightarrow n = \infty$

$V(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-nI}$

replace 'n' by 'm'

$V(z) = \sum_{m=-\infty}^{\infty} x(m) \cdot z^{-mI}$

For Frequency response, put $z = e^{j\omega_y}$

$$\Rightarrow V\left(e^{j\omega_y}\right) = X\left(e^{(j\omega_y)I}\right)$$

$$= X\left(e^{j(\omega_y I)}\right)$$

$$\Rightarrow \boxed{V(\omega_y) = X(\omega_y I)} \quad \text{--- (1)}$$

Spectrum calculation for o/p of up sampler

The above equation represents the frequency domain representation of interpolation system.

Frequency range calculation for 'ω_y':-

WkT, relative frequency ω_y w.r.to F_y is

$$\omega_y = \frac{2\pi F_{max}}{F_y}$$

Similarly, $\omega_x = \frac{2\pi F_{max}}{F_x}$

But output of sampler frequency, $F_y = I F_x$

$$\Rightarrow \quad \frac{2\pi F_{max}}{\omega_y} = I\left(\frac{2\pi F_{max}}{\omega_x}\right)$$

$$\Rightarrow \quad \frac{1}{\omega_y} = \frac{I}{\omega_x}$$

$$\Rightarrow \quad \omega_y = \frac{\omega_x}{I}$$

WkT, Input signal frequency range is: $-\pi \leq \omega_x \leq \pi$

$$\Rightarrow \quad \frac{-\pi}{I} \leq \frac{\omega_x}{I} \leq \frac{\pi}{I}$$

$$\Rightarrow \quad \boxed{\frac{-\pi}{I} \leq \omega_y \leq \frac{\pi}{I}}$$

Frequency range calculation for 'ω_y'.

$\therefore$ The range of ω_y is: $\frac{-\pi}{I} \leq \omega_y \leq \frac{\pi}{I}$

→ The frequency response of anti-imaging filter or Low pass filter is given by:

$$H(\omega_y) = \begin{cases} C \; ; & \frac{-\pi}{I} \leq \omega_y \leq \frac{\pi}{I} \\ 0 \; ; & \text{otherwise} \end{cases} \quad \text{(2)}$$

where, C is scaling factor.

For normalization, put $C = I$

→ Output spectrum of anti-imaging filter is:

$$Y(\omega_y) = \begin{cases} C\,X(\omega_y I) \;:\; -\frac{\pi}{I} \le \omega_y \le \frac{\pi}{I} \\ 0 \;:\; \text{Otherwise} \end{cases}$$

$y(m) = v(m) * h(m)$

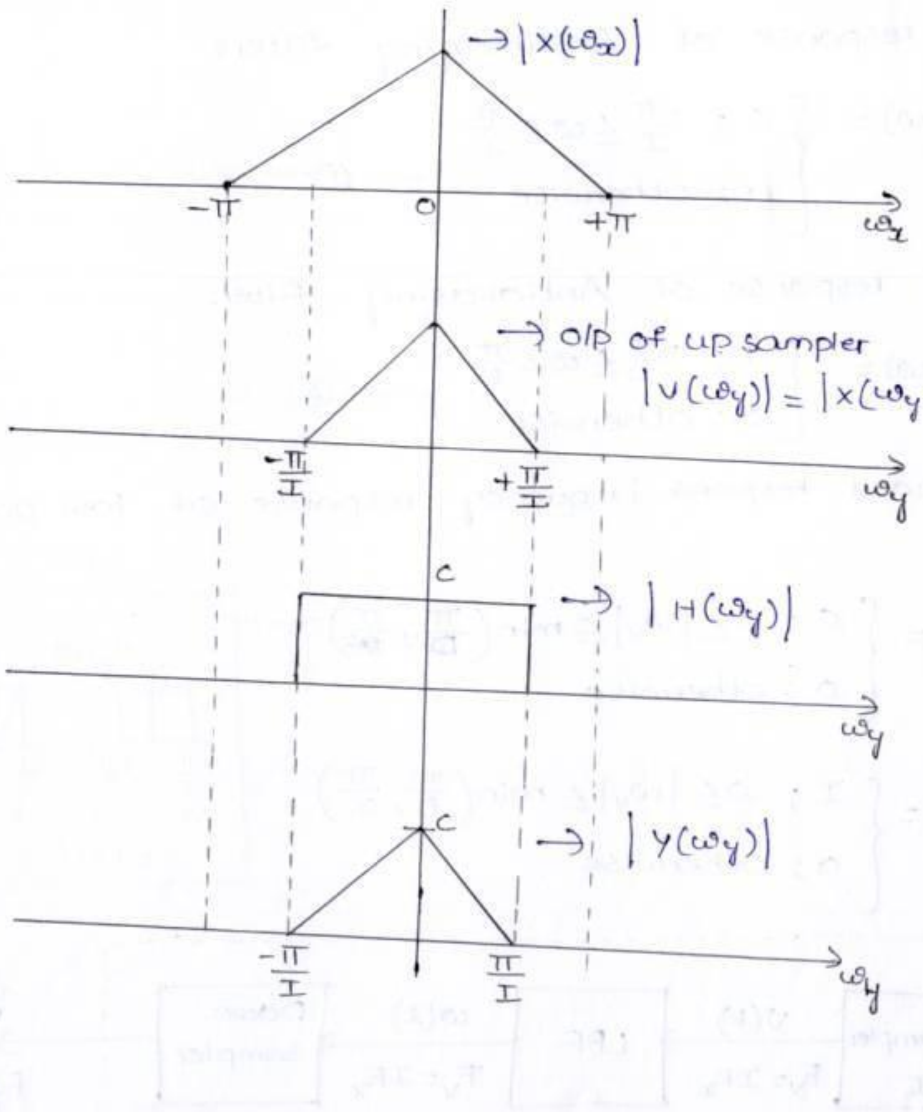
$\Rightarrow Y(Z) = V(Z) \cdot H(Z)$

$\Rightarrow Y(e^{j\omega_y}) = V(e^{j\omega_y})\, H(e^{j\omega_y})$

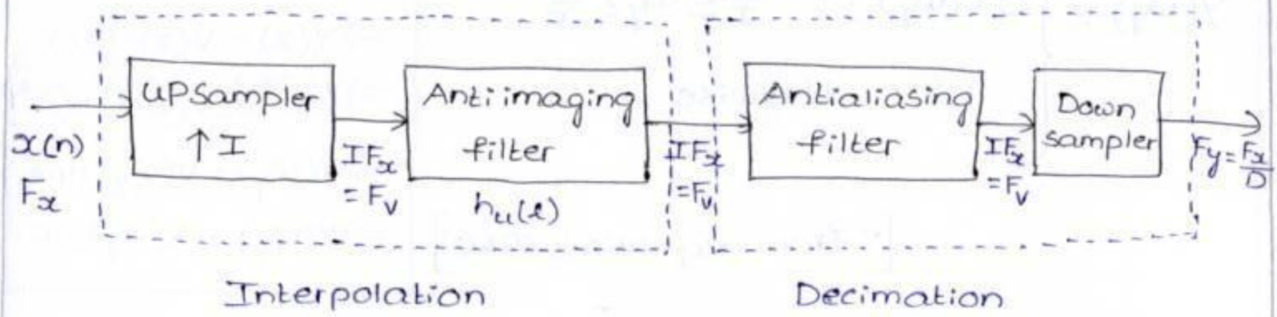
$\Rightarrow Y(\omega_y) = V(\omega_y)\, H(\omega_y)$

$\Rightarrow Y(\omega_y) = X(\omega_y I) \cdot C$

[∵ from equation's ① & ②]



→ Sampling rate conversion by non-integer factor (or) Rational number I/D :-



→ Frequency response of Antiimaging filter:

$$H_u(\omega) = \begin{cases} C ; & \frac{-\pi}{I} \leq \omega \leq \frac{\pi}{I} \\ 0 : & \text{Otherwise} \end{cases} \quad \text{—— ①}$$

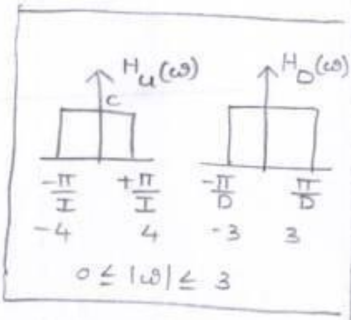
→ Frequency response of Antialiasing filter:

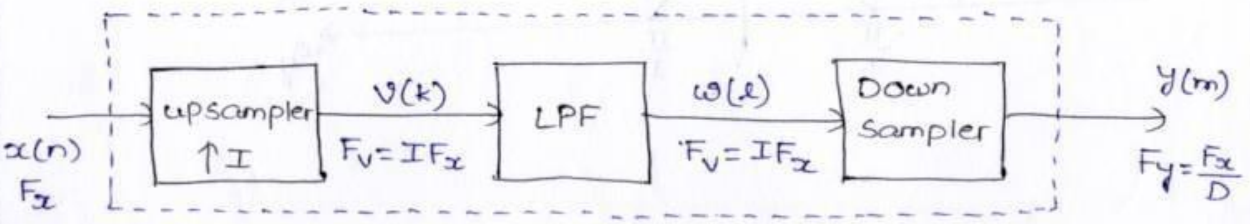
$$H_d(\omega) = \begin{cases} 1 ; & -\frac{\pi}{D} \leq \omega < \frac{\pi}{D} \\ 0 ; & \text{otherwise} \end{cases} \quad \text{—— ②}$$

→ The cascaded ~~respons~~ Frequency response of low pass filter:

$$H(\omega) = \begin{cases} C ; & 0 \leq |\omega_v| \leq \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0 ; & \text{otherwise} \end{cases}$$

$$= \begin{cases} I ; & 0 \leq |\omega_v| \leq \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0 ; & \text{Otherwise} \end{cases}$$





→ output of upsampler:

$$v(k) = \begin{cases} x\left(\frac{k}{I}\right) & ; k = 0, \pm I, \pm 2I, \cdots\cdots \\ 0 & ; \text{otherwise} \end{cases} \quad \text{—— ③}$$

→ Output of LPF:

$$\omega(l) = v(l) * h(l)$$
$$= \sum_{k=-\infty}^{\infty} v(k) \cdot h(l-k)$$

On substituting eq-③ in above equation, we have

$$\Rightarrow \omega(l) = \sum_{k=-\infty}^{\infty} x\left(\frac{k}{I}\right) \cdot h(l-k)$$

Replace 'k' by 'kI', we have

$$\omega(l) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(l-kI) \quad \text{—— } ④$$

→ Output of downsampler:

$$y(m) = \omega(mD)$$
$$= \sum_{k=-\infty}^{\infty} x(k) \cdot h(mD-kI) \text{ — } ⑤ \left[\because \text{From eq-}④\right]$$

Let $\left\lfloor \frac{mD}{I} \right\rfloor - n = k$ —— ⑥

where, $\lfloor r \rfloor \Rightarrow$ produces largest integer present in 'r'.

On substituting equation-⑥ in eq-⑤, we have

$$\Rightarrow y(m) = \sum_{n=-\infty}^{\infty} x\left(\left\lfloor \frac{mD}{I} \right\rfloor - n\right) h\left[mD - \left(\left\lfloor \frac{mD}{I} \right\rfloor - n\right) I\right]$$
$$= \sum_{n=-\infty}^{\infty} x\left(\left\lfloor \frac{mD}{I} \right\rfloor - n\right) h\left[\underbrace{mD - \left\lfloor \frac{mD}{I} \right\rfloor I}_{(mD)_I} + nI\right]$$

WKT, $mD - \left\lfloor \frac{mD}{I} \right\rfloor * I = mD$, modulo $I = (mD)_I$

$$\boxed{y(m) = \sum_{n=-\infty}^{\infty} x\left(\left\lfloor \frac{mD}{I} \right\rfloor - n\right) h\left[(mD)_I + nI\right]} \quad \text{—— (7)}$$

The above expression represents time domain equation for conversion factor $\frac{I}{D}$.

→ Output of down sampler spectrum: (direct result from Decimation i.e., down sampler)

$$Y(\omega_y) = \begin{cases} \frac{1}{D} X\left(\frac{\omega_y}{D}\right) & ; 0 \le \omega_y \le \min\left(\pi, \frac{\pi D}{I}\right) \\ 0 & ; \text{otherwise} \end{cases}$$

WKT,

$$\left[\because \omega_y = D\,\omega_v \right.$$

$$\Rightarrow \omega_y = D\left[0 \le |\omega_v| \le \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right)\right]$$

$$\Rightarrow \omega_y = 0 \le |\omega_v D| \le \min\left(\pi, \frac{\pi D}{I}\right)$$

→ Determine sampling rate conversion by a factor of $\frac{5}{3}$ for the following sequence:

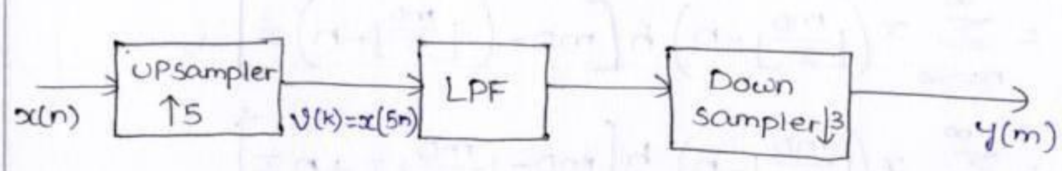
$x(n) = \{3, 1, 2, 4\}$

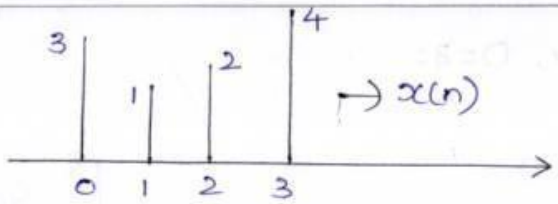
Sol: Given, $x(n) = \{3, 1, 2, 4\}$

Factor, $\frac{I}{D} = \frac{5}{3}$

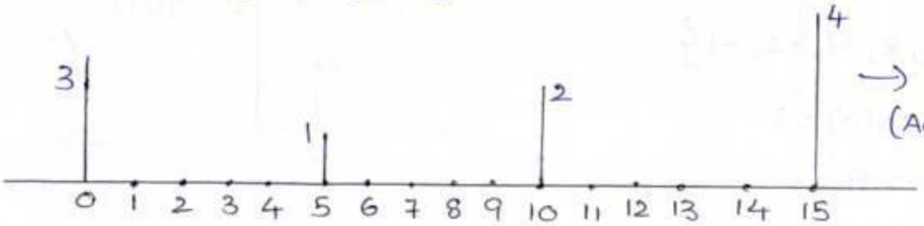
$\therefore I = 5$

$D = 3$

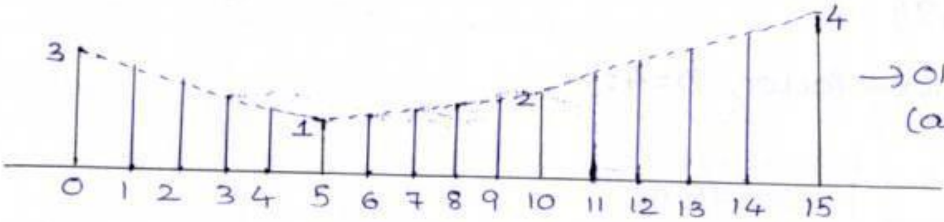




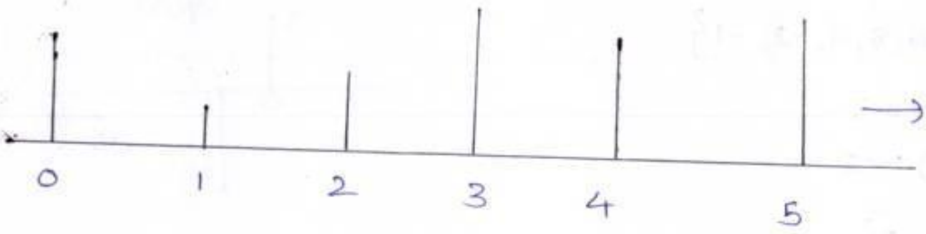
$\rightarrow x(n)$



→ O/P of upsampler (Add (5-1) zero's in b/w samples)



→ O/P of LPF (anti imaging filter)

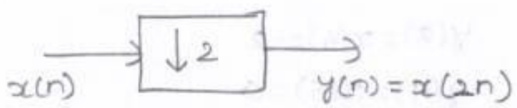


→ O/P of down sampler (every Dth sample i.e., 3rd sample of sequence)

→ Find Decimated and interpolated sequence of following signal, $x(n) = \{3, 6, 8, 9, -2, -1\}$ with factors 2, 3, 4

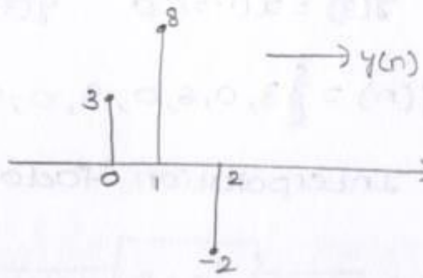
Sol: Case-1:-

For decimation factor, D=2



$\rightarrow x(n) = \{3, 6, 8, 9, -2, -1\}$

$\rightarrow y(n) = \{3, 8, -2\}$



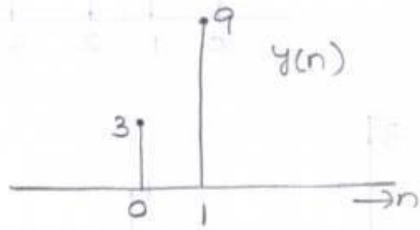
$\therefore y(0) = x(0)$

$y(1) = x(2(1)) = x(2)$

$y(2) = x(4)$

For decimation factor, $D=3$:

$$x(n) \longrightarrow \boxed{\downarrow 3} \longrightarrow y(n) = x(3n)$$



$x(n) = \{3, 6, 8, 9, -2, -1\}$

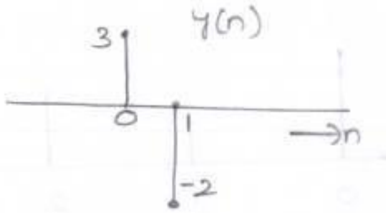
$y(0) = \cancel{y(0)}\ x(0) = 3$

$y(1) = x(3) = 9$

$\therefore y(n) = \{3, 9\}$

For decimation factor, $D=4$:

$$x(n) \longrightarrow \boxed{\downarrow 4} \longrightarrow y(n) = x(4n)$$



$x(n) = \{3, 6, 8, 9, -2, -1\}$

$y(0) = x(0) = 3$

$y(1) = x(4) = -2$

$\therefore y(n) = \{3, -2\}$

Case-(ii):

For Interpolation factor, $I=2$:

$$x(n) \longrightarrow \boxed{\uparrow I=2} \longrightarrow y(n) = x\left(\frac{n}{I}\right) = x\left(\frac{n}{2}\right)$$

$x(n) = \{3, 6, 8, 9, -2, -1\}$

$\cancel{x(0)}\ y(0) = x(0) = 3 \qquad y(4) = x(2) = 8 \qquad y(8) = x(4) = -2$

$y(1) = x(0.5) = 0 \qquad y(5) = x(2.5) = 0 \qquad y(9) = x(5) = 0$

$y(2) = x(1) = 6 \qquad y(6) = x(3) = 9 \qquad y(10) = x(6) = -1$

$y(3) = x(1.5) = 0 \qquad y(7) = x(3.5) = 0$

$\therefore y(n) = \{3, 0, 6, 0, 8, 0, 9, 0, -2, 0, -1\}$

For Interpolation factor, $I=3$:

$$x(n) \longrightarrow \boxed{\uparrow I=3} \longrightarrow y(n) = x\left(\frac{n}{I}\right) = x\left(\frac{n}{3}\right)$$

$x(n) = \{3, 6, 8, 9, -2, -1\}$

$(3-1) = 2$ zero's are to be inserted b/w values to get $y(n)$ sequence.

$\therefore y(n) = \{3, 0, 0, 6, 0, 0, 8, 0, 0, 9, 0, 0, -2, 0, 0, -1\}$

For interpolation factor, I=4:

$$x(n) \longrightarrow \boxed{\uparrow I = 4} \longrightarrow y(n) = x\left(\frac{n}{4}\right)$$

$x(n) = \{3, 6, 8, 9, -2, -1\}$

$(4-1) = 3$ zero's are to be inserted b/w values to get $y(n)$ sequence.

$\therefore y(n) = \{3, 0, 0, 0, 6, 0, 0, 0, 8, 0, 0, 0, 9, 0, 0, 0, -2, 0, 0, 0, -1\}$

→ Consider a sequence $x(n) = a^n \cdot u(n)$

(i) Determine the spectrum of the signal

(ii) The signal is applied to a decimator that reduces by a sampling factor of 2. Draw the o/p spectrum.

(iii) The signal is applied to a interpolator that increases by a factor of '2'. Determine the o/p spectrum.

(iv) Show that the spectrum in part 'iii' is simply fourier transform of $x(n/2)$.

Sol: Given, $x(n) = a^n \cdot u(n)$

(i)

WKT,

$$x(n) = a^n u(n) \longrightarrow \boxed{S/m} \longrightarrow X(\omega)$$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$Z\{a^n \cdot u(n)\} = \sum_{n=-\infty}^{\infty} a^n \cdot u(n)\, z^{-n}$$

$$\boxed{\begin{aligned} Z\{x(n)\} &= \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} \\ Z\{a^n \cdot u(n)\} &= \sum_{n=-\infty}^{\infty} a^n \cdot u(n)\, z^{-n} \end{aligned}}$$

$$= \sum_{n=0}^{\infty} a^n (1) z^{-n} \qquad \left[\because \sum_{n=0}^{\infty} r^n = 1 + r^1 + r^2 + r^3 + \cdots\cdots \right.$$

$$= \sum_{n=0}^{\infty} (a \cdot z^{-1})^n \qquad = (1-r)^{-1}$$

$$\left. = \frac{1}{1-r} \ ; |r|<1 \right]$$

$$= \frac{1}{1-az^{-1}}$$

$$= \frac{1}{1-\frac{a}{z}}$$

$$z\{a^n \cdot u(n)\} = \frac{z}{z-a}$$

$$\therefore \quad z\{a^n \cdot u(n)\} = \frac{z}{z-a} \ ; \ |az^{-1}|<1$$

$$\Rightarrow \left|\frac{a}{z}\right| < 1$$

$$\boxed{X(z) = \frac{z}{z-a}} \qquad \Rightarrow |z| > a$$

ROA

For frequency domain, put $z = e^{j\omega}$

$$\Rightarrow X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}$$

$$\Rightarrow \quad X(\omega) = \frac{\cos\omega + j\sin\omega}{\cos\omega + j\sin\omega - a}$$

$$\Rightarrow \quad |X(\omega)| = \left| \frac{\cos\omega + j\sin\omega}{(\cos\omega - a) + j\sin\omega} \right|$$

$$= \frac{|\cos\omega + j\sin\omega|}{|(\cos\omega - a) + j\sin\omega|}$$

$$= \frac{\sqrt{\cos^2\omega + \sin^2\omega}}{\sqrt{(\cos\omega - a)^2 + \sin^2\omega}}$$

$$\Rightarrow |X(\omega)| = \frac{1}{\sqrt{(\cos\omega - a)^2 + \sin^2\omega}}$$

$$= \frac{1}{\sqrt{\cos^2\omega - 2a\cos\omega + a^2 + \sin^2\omega}}$$

$$= \frac{1}{\sqrt{a^2 - 2a\cos\omega + 1}}$$

$$= \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

At:

$$\omega = 0 \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^2-2a}} = \frac{1}{\sqrt{(1-a)^2}} = \frac{1}{1-a}$$

$$\omega = \frac{\pi}{4} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^2-2a(\frac{1}{\sqrt{2}})}} = \frac{1}{\sqrt{1+a^2-\sqrt{2}a}}$$

$$\omega = \frac{\pi}{2} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^2-0}} = \frac{1}{\sqrt{1+a^2}}$$

$$\omega = \frac{3\pi}{4} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^2-2a(-\frac{1}{\sqrt{2}})}} = \frac{1}{\sqrt{1+a^2+\sqrt{2}a}}$$

$$\omega = \pi \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^2-2a(-1)}} = \frac{1}{\sqrt{1+a^2+2a}} = \frac{1}{\sqrt{(1+a)^2}} = \frac{1}{1+a}$$

$$\omega = \frac{5\pi}{4} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^2-2a(\frac{-1}{\sqrt{2}})}} = \frac{1}{\sqrt{1+a^2+\sqrt{2}a}}$$

$$\omega = \frac{3\pi}{2} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^2-2a(0)}} = \frac{1}{\sqrt{1+a^2}}$$

$$\omega = \frac{7\pi}{4} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^2-2a(\frac{1}{\sqrt{2}})}} = \frac{1}{\sqrt{1+a^2-\sqrt{2}a}}$$

$$\omega = \pi \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^2-2a}} = \frac{1}{\sqrt{(1-a)^2}} = \frac{1}{1-a}$$

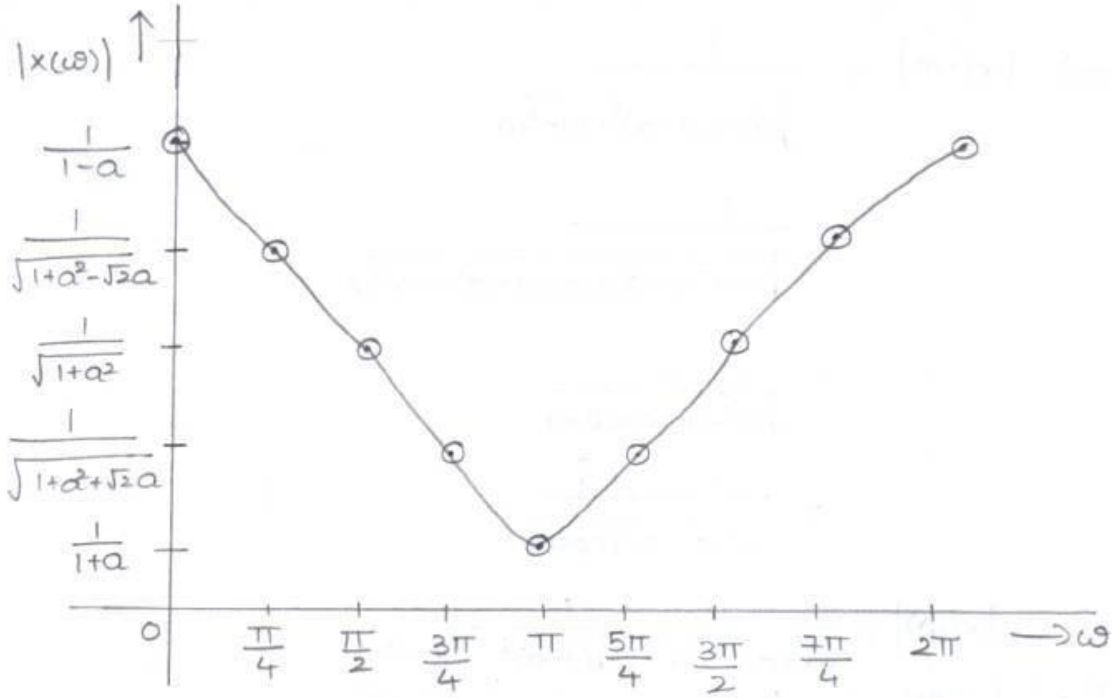


Fig: Magnitude spectrum

(ii)

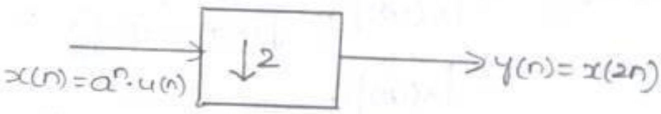


Fig: Decimator

$$y(n) = x(2n) = a^{2n}u(2n)$$

$$Z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n)\cdot z^{-n}$$

$$\Rightarrow Z\{a^{2n}u(2n)\} = \sum_{n=-\infty}^{\infty}\left[a^{2n}u(2n)\right]z^{-n}$$

$$= \sum_{n=0}^{\infty}\left[a^{2n}(1)\right]z^{-n}$$

$$= \sum_{n=0}^{\infty} a^{2n}z^{-n}$$

$$= \sum_{n=0}^{\infty}\left(a^2z^{-1}\right)^{+n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{a^2}{z}\right)^{n}$$

$$= \cancel{\sum_{n=0}^{\infty}}\ \frac{1}{1-\frac{a^2}{z}}$$

$$u(n) = \begin{cases} 1; & n \geq 0 \\ 0; & n < 0 \end{cases}$$

$$\Rightarrow u(2n) = \begin{cases} 1; & 2n \geq 0 \Rightarrow n \geq 0 \\ 0; & 2n < 0 \Rightarrow n < 0 \end{cases}$$

$$\Rightarrow u(2n) = \begin{cases} 1; & n \geq 0 \\ 0; & n < 0 \end{cases}$$

$$\left[\because \sum_{n=0}^{\infty} r^n = \frac{1}{1-r};\ |r|<1\right]$$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} \frac{z}{z-a^2}$$

$$\boxed{X(z) = \frac{z}{z-a^2}} \; : \left|\frac{a^2}{z}\right| < 1$$

$$\Rightarrow |a^2| < z$$

$$\Rightarrow |z| > a^2$$

$\left[\because |z| = \sqrt{(a^2)^2}\right.$

For frequency domain, put $z = e^{j\omega}$

$$\Rightarrow X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a^2}$$

$$\Rightarrow X(e^{j\omega}) = \frac{\cos\omega + j\sin\omega}{\cos\omega + j\sin\omega - a^2}$$

$$\Rightarrow |X(e^{j\omega})| = \left|\frac{\cos\omega + j\sin\omega}{(\cos\omega - a^2) + j\sin\omega}\right|$$

$$= \frac{|\cos\omega + j\sin\omega|}{|(\cos\omega - a^2) + j\sin\omega|}$$

$$= \frac{1}{\sqrt{(\cos\omega - a^2)^2 + \sin^2\omega}}$$

$$= \frac{1}{\sqrt{\cos^2\omega - 2a^2\cos\omega + a^4 + \sin^2\omega}}$$

$$= \frac{1}{\sqrt{1 + a^4 - 2a^2\cos\omega}}$$

$$\omega = 0 \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^4-2a^2(1)}} = \frac{1}{\sqrt{1+a^4-2a^2}} = \frac{1}{1-a^2}$$

$$\omega = \frac{\pi}{4} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^4-2a^2\left(\frac{1}{\sqrt{2}}\right)}} = \frac{1}{\sqrt{1+a^4-\sqrt{2}a^2}}$$

$$\omega = \frac{\pi}{2} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^4-2a^2(0)}} = \frac{1}{\sqrt{1+a^4}}$$

$$\omega = \frac{3\pi}{4} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^4-2a^2\left(-\frac{1}{\sqrt{2}}\right)}} = \frac{1}{\sqrt{1+a^4+\sqrt{2}a^2}}$$

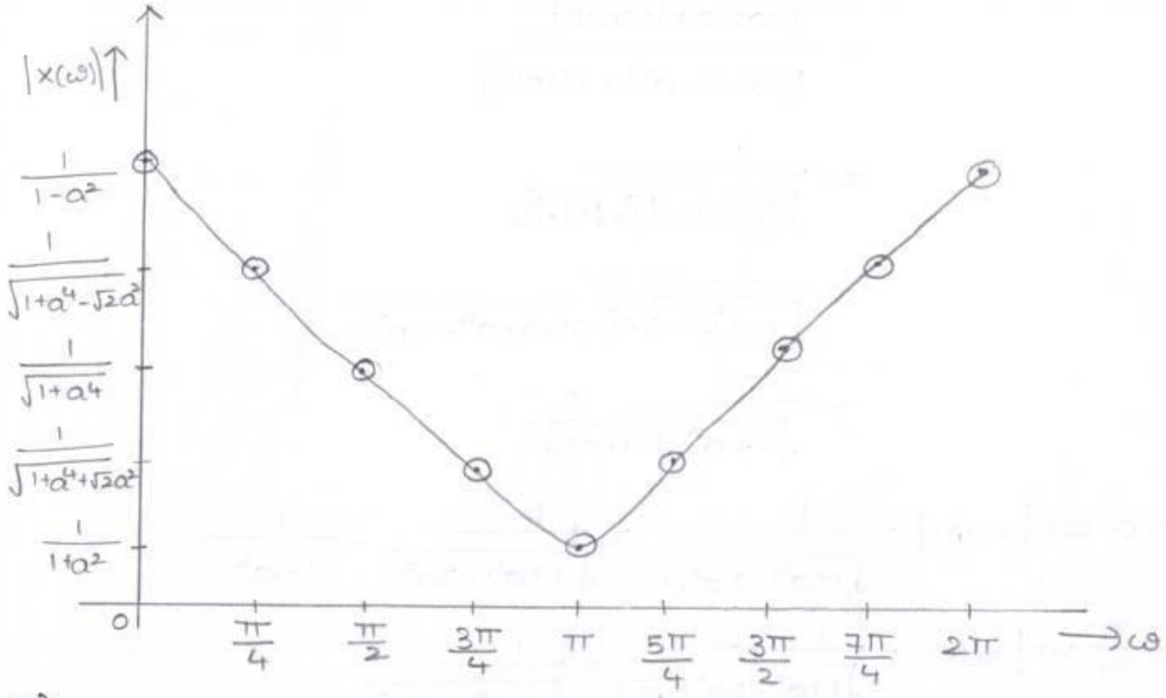
$$\omega = \pi \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^4-2a^2(-1)}} = \frac{1}{\sqrt{1+a^4+2a^2}} = \frac{1}{1+a^2}$$

$$\omega = \frac{5\pi}{4} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^4-2a^2\left(\frac{-1}{\sqrt{2}}\right)}} = \frac{1}{\sqrt{1+a^4+\sqrt{2}a^2}}$$

$$\omega = \frac{3\pi}{2} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^4-2a^2(0)}} = \frac{1}{\sqrt{1+a^4}}$$

$$\omega = \frac{7\pi}{4} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^4-2a^2\left(\frac{1}{\sqrt{2}}\right)}} = \frac{1}{\sqrt{1+a^4-\sqrt{2}a^2}}$$

$$\omega = \pi \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a^4-2a^2}} = \frac{1}{\sqrt{(1-a^2)^2}} = \frac{1}{1-a^2}$$



(iii)

Given, $x(n) = a^n \cdot u(n)$

$$y(n) = x\left(\frac{n}{2}\right) = a^{\frac{n}{2}} \cdot u\left(\frac{n}{2}\right)$$

$$Z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n) \cdot z^{-n}$$

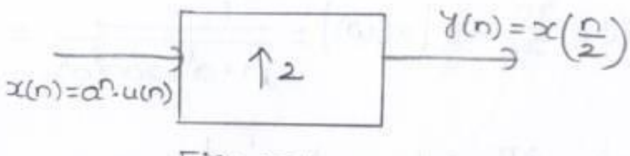


Fig: Interpolator

$$\Rightarrow Z\{a^n \cdot u(n)\} = \sum_{n=-\infty}^{\infty} a^n \cdot u(n) \cdot z^{-n}$$

$$\Rightarrow Z\{a^{n/2} \cdot u(\tfrac{n}{2})\} = \sum_{n=-\infty}^{\infty} a^{n/2} u(\tfrac{n}{2}) \cdot z^{-n}$$

$$\left[\because u(n) = \begin{cases} 1; & n \geq 0 \\ 0; & n < 0 \end{cases}\right.$$

$$u(\tfrac{n}{2}) = \begin{cases} 1; & \tfrac{n}{2} \geq 0 \Rightarrow n \geq 0 \\ 0; & \tfrac{n}{2} < 0 \Rightarrow n < 0 \end{cases}$$

$$= \sum_{n=0}^{\infty} a^{n/2} (1) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} (a^{+1/2} \cdot z^{-1})^n$$

$$= \sum_{n=0}^{\infty} (a^{1/2} \cdot z^{-1})^n \qquad \left[\because \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}; |r| < 1\right.$$

$$= \sum_{n=0}^{\infty} \left(\frac{a^{1/2}}{z}\right)^n$$

$$= \frac{1}{1-\left(\frac{a^{1/2}}{z}\right)}$$

$$X(z) = \frac{z}{z-a^{1/2}} \quad ; \quad \left|\frac{a^{1/2}}{z}\right| < 1$$

$$\Rightarrow |a^{1/2}| < z$$

$$\Rightarrow z > |a^{1/2}|$$

For frequency domain, put $z = e^{j\omega}$

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a^{1/2}}$$

$$\Rightarrow X(\omega) = \frac{\cos\omega + j\sin\omega}{\cos\omega + j\sin\omega - a^{1/2}}$$

$$\Rightarrow |X(\omega)| = \left|\frac{\cos\omega + j\sin\omega}{(\cos\omega - a^{1/2}) + j\sin\omega}\right|$$

$$\Rightarrow |X(\omega)| = \frac{1}{\sqrt{(\cos\omega - a^{1/2})^2 + \sin^2\omega}}$$

$$= \frac{1}{\sqrt{\cos^2\omega - 2\cos\omega\, a^{1/2} + a + \sin^2\omega}}$$

$$|X(\omega)| = \frac{1}{\sqrt{1 + a - 2\cos\omega\, a^{1/2}}}$$

At

$$\omega = 0 \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a-2a^{1/2}}}$$

$$\omega = \frac{\pi}{4} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a-2a^{1/2}\left(\frac{1}{\sqrt{2}}\right)}} = \frac{1}{\sqrt{1+a-\sqrt{2}a^{1/2}}}$$

$$\omega = \frac{\pi}{2} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a-2a^{1/2}(0)}} = \frac{1}{\sqrt{1+a}}$$

$$\omega = \frac{3\pi}{4} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a-2a^{1/2}\left(\frac{-1}{\sqrt{2}}\right)}} = \frac{1}{\sqrt{1+a+\sqrt{2}a^{1/2}}}$$

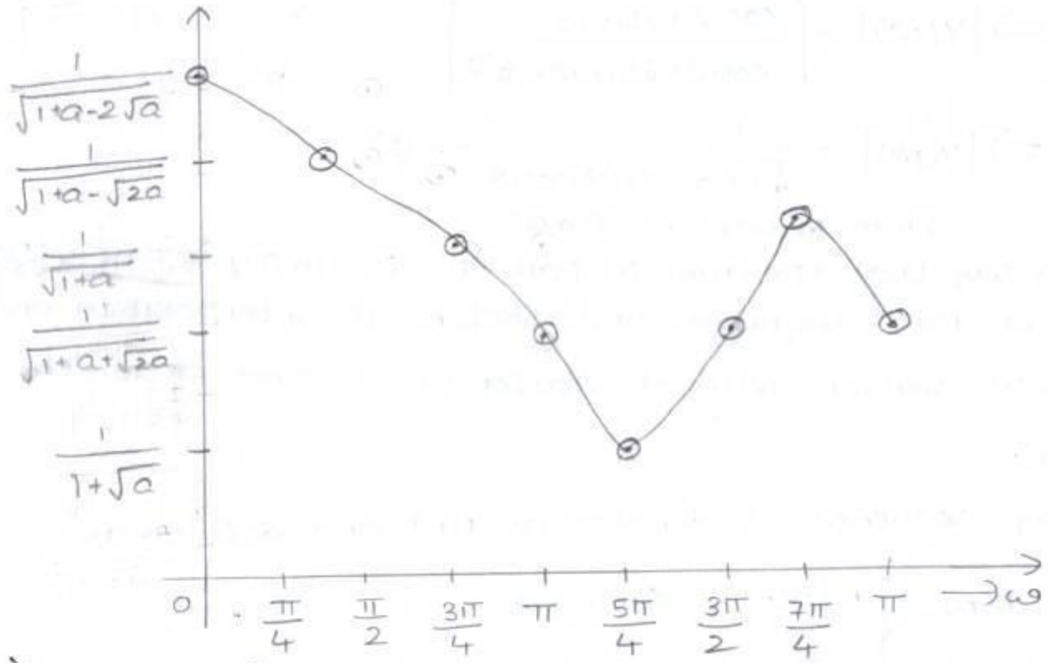
$$\omega = \pi \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a-2a^{1/2}(-1)}} = \frac{1}{\sqrt{1+a+2a^{1/2}}} = \frac{1}{1+a^{1/2}}$$

$$\omega = \frac{5\pi}{4} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a-2a^{1/2}\left(\frac{-1}{\sqrt{2}}\right)}} = \frac{1}{\sqrt{1+a+\sqrt{2}a^{1/2}}}$$

$$\omega = \frac{3\pi}{2} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a-2a^{1/2}(0)}} = \frac{1}{\sqrt{1+a}}$$

$$\omega = \frac{7\pi}{4} \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a-2a^{1/2}\left(\frac{1}{\sqrt{2}}\right)}} = \frac{1}{\sqrt{1+a-\sqrt{2}a^{1/2}}} = \frac{1}{\sqrt{1+a-\sqrt{2a}}}$$

$$\omega = \pi \Rightarrow |X(\omega)| = \frac{1}{\sqrt{1+a-2(-1)a^{1/2}}} = \frac{1}{\sqrt{1+a+2\sqrt{a}}}$$



(iv):

In part(c), equation for output spectrum is given by

$$|Y(\omega)| = \frac{1}{\sqrt{1+a-2a^{1/2}\cos\omega}}$$

$$\text{DTFT}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\cdot e^{-j\omega n} \quad \text{———} \quad (1)$$

$$\Rightarrow \text{DTFT}\left\{x\left(\frac{n}{2}\right)\right\} = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2}\right)\cdot e^{-j\omega \frac{n}{2}}$$

$$\left[\because x\left(\frac{n}{2}\right) = a^{n/2}\cdot u(n/2)\right]$$

$$= \sum_{n=-\infty}^{\infty} \left[a^{n/2}\cdot u^{n/2}\right]\cdot e^{\frac{-j\omega n}{2}}$$

$$= \sum_{n=0}^{\infty} a^{n/2}\cdot e^{\frac{-j\omega n}{2}}$$

$$= \sum_{n=0}^{\infty} \left(a^{1/2}\cdot e^{-j\omega}\right)^n$$

$$\left[\because \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}\right]$$

$$= \frac{1}{1-\frac{a^{1/2}}{e^{j\omega}}}$$

$$= \frac{e^{j\omega}}{e^{j\omega} - a^{1/2}}$$

$$\Rightarrow |Y(\omega)| = \left| \frac{\cos\omega + j\sin\omega}{\cos\omega + j\sin\omega - a^{1/2}} \right|$$

$$\Rightarrow |Y(\omega)| = \frac{1}{\sqrt{1 + a - 2a^{1/2}\cos\omega}} \quad \text{— (2)}$$

From equations (1) & (2)

We can say that spectrum in part-(iii) is simply FT of $x\left[\frac{n}{2}\right]$.

→ Define a ramp sequence and sketch its interpolation and decimation version with a factor of 3 and draw the Spectrum.

Sol: The ramp sequence is denoted as $r(n)$ and defined as

$$r(n) = \begin{cases} n \cdot u(n), & \text{for } n \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

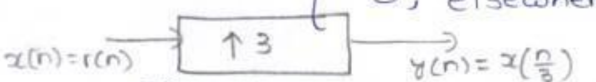


Fig: Interpolator

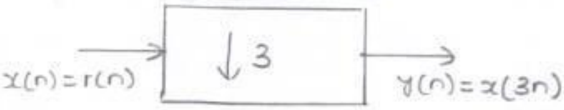
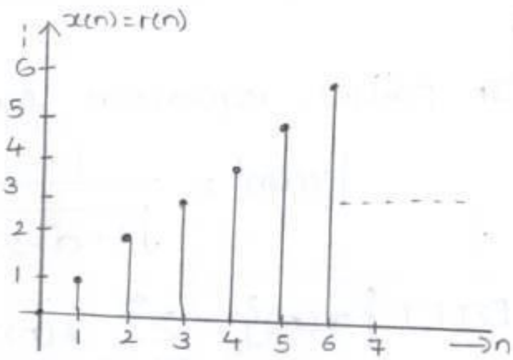


Fig: Decimator



(a) Fig: Ramp sequence

→ The decimated sequence can be obtained by selecting every Dth sample of input sequence shown in fig (b).

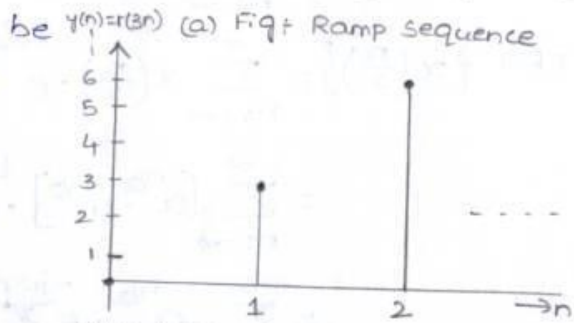


Fig (b): Decimated ramp sequence

→ The interpolated ramp sequence can be obtained by inserting two Zero's between every two Successive sampling instants shown in fig (c).

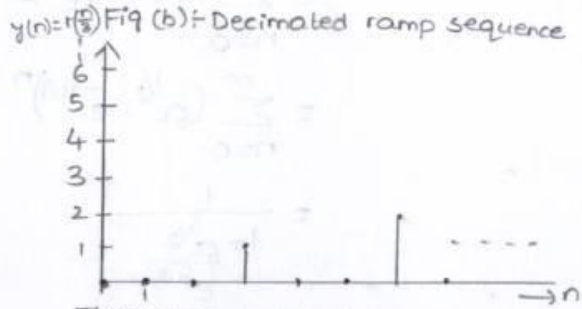


Fig (c): Interpolated ramp sequence

Output spectrum of decimator:

From the figure of decimator, $y(n) = x(3n)$

$$= 3n \cdot u(3n)$$

$$Z\{y(n)\} = Y(z) = \sum_{n=-\infty}^{\infty} y(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 3n \cdot u(3n) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} 3n \cdot (1)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} 3n \cdot z^{-n}$$

$$= 3\sum_{n=0}^{\infty} n \cdot (z^{-1})^{n}$$

$$= 3\left[z^{-1}\left[1+2z^{-1}+3z^{-2}+\cdots\right]\right]$$

$$= 3z^{-1}\left[1-z^{-1}\right]^{-2}$$

$$= 3z^{-1}\left[\frac{1}{\left(1-\frac{1}{z}\right)^2}\right]$$

$$= 3z^{-1}\left[\frac{1}{\left(\frac{z-1}{z}\right)^2}\right]$$

$$= \frac{3}{z} \cdot \frac{z^2}{(z-1)^2}$$

$$= \frac{3z}{(z-1)^2}$$

For frequency domain, put $z = e^{j\omega}$

$$\Rightarrow Y(e^{j\omega}) = \frac{3e^{j\omega}}{(e^{j\omega}-1)^2}$$

The magnitude spectrum $Y(\omega)$ of o/p sequence $y(n)$ is

$$|Y(\omega)| = \frac{3}{\sqrt{6-8\cos\omega+2\cos 2\omega}}$$

At $\omega=0$, $|Y(\omega)| = \dfrac{3}{\sqrt{6-8+2}} = \infty$

At $\omega = \frac{\pi}{4} \Rightarrow |Y(\omega)| = \frac{3}{\sqrt{6-8\cos\frac{\pi}{4}+2\cos 2\left(\frac{\pi}{4}\right)}} = 5.1213$

At $\omega = \frac{\pi}{2} \Rightarrow |Y(\omega)| = \frac{3}{\sqrt{6-8\cos\left(\frac{\pi}{2}\right)+2\cos 2\left(\frac{\pi}{2}\right)}} = 1.5$

At $\omega = \frac{3\pi}{4} \Rightarrow |Y(\omega)| = \frac{3}{\sqrt{6-8\cos\left(\frac{3\pi}{4}\right)+2\cos 2\left(\frac{3\pi}{4}\right)}} = 0.879$

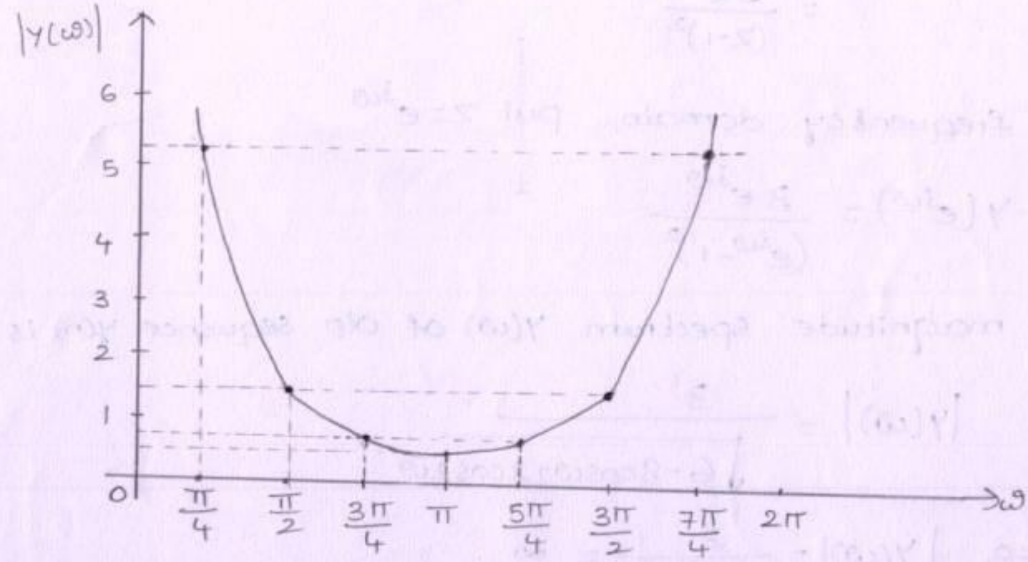
At $\omega = \pi \Rightarrow |Y(\omega)| = \frac{3}{\sqrt{6-8\cos\pi+2\cos 2\pi}} = 0.75$

At $\omega = \frac{5\pi}{4} \Rightarrow |Y(\omega)| = \frac{3}{\sqrt{6-8\cos\left(\frac{5\pi}{4}\right)+2\cos 2\left(\frac{5\pi}{4}\right)}} = 0.879$

At $\omega = \frac{3\pi}{2} \Rightarrow |Y(\omega)| = \frac{3}{\sqrt{6-8\cos\left(\frac{3\pi}{2}\right)+2\cos 2\left(\frac{3\pi}{2}\right)}} = 1.5$

At $\omega = \frac{7\pi}{4} \Rightarrow |Y(\omega)| = \frac{3}{\sqrt{6-8\cos\left(\frac{7\pi}{4}\right)+2\cos 2\left(\frac{7\pi}{4}\right)}} = 5.121$

At $\omega = 2\pi \Rightarrow |Y(\omega)| = \frac{3}{\sqrt{6-8\cos 2\pi+2\cos 4\pi}} = \infty$



Output Spectrum of Interpolator:

From the figure of interpolator, $y(n) = x\left(\frac{n}{3}\right)$

$$= \frac{n}{3} \cdot u\left(\frac{n}{3}\right)$$

$$Z\{y(n)\} = Y(z) = \sum_{n=-\infty}^{\infty} y(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{n}{3} \cdot u\left(\frac{n}{3}\right) z^{-n}$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} n \cdot u\left(\frac{n}{3}\right) \cdot z^{-n}$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} n \cdot (1) z^{-n}$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} n \cdot z^{-n}$$

$$= \frac{1}{3}\left[z^{-1}\left[1 + 2z^{-1} + 3z^{-2} + \ldots\right]\right]$$

$$= \frac{1}{3} z^{-1}\left[1 - z^{-1}\right]^{-2}$$

$$= \frac{1}{3} z^{-1}\left[\frac{1}{(1-z^{-1})^2}\right]$$

$$= \frac{1}{3} z^{-1}\left[\frac{1}{\left(\frac{z-1}{z}\right)^2}\right]$$

$$= \frac{1}{3\cancel{z}} \cdot \frac{z^{\cancel{2}}}{(z-1)^2}$$

$$= \frac{z}{3(z-1)^2}$$

For frequency domain, put $z = e^{j\omega}$

$$\Rightarrow Y(e^{j\omega}) = \frac{1}{3} \cdot \frac{e^{j\omega}}{(e^{j\omega} - 1)^2}$$

The magnitude spectrum $Y(\omega)$ of o/p sequence $y(n)$ is,

$$|Y(\omega)| = \frac{1}{3} \cdot \frac{1}{\sqrt{6 - 8\cos\omega + 2\cos 2\omega}}$$

At $\omega = 0$, $|Y(\omega)| = \frac{1}{3\sqrt{6-8+2}} = \infty$

At $\omega = \frac{\pi}{4} \Rightarrow |Y(\omega)| = \frac{1}{3} \cdot \frac{1}{\sqrt{6-8\cos\left(\frac{\pi}{4}\right)+2\cos 2\left(\frac{\pi}{4}\right)}} = 0.56$

At $\omega = \frac{\pi}{2} \Rightarrow |Y(\omega)| = \frac{1}{3} \cdot \frac{1}{\sqrt{6-8\cos\left(\frac{\pi}{2}\right)+2\cos 2\left(\frac{\pi}{2}\right)}} = 0.16$

At $\omega = \frac{3\pi}{4} \Rightarrow |Y(\omega)| = \frac{1}{3} \cdot \frac{1}{\sqrt{6-8\cos\left(\frac{3\pi}{4}\right)+2\cos 2\left(\frac{3\pi}{4}\right)}} = 0.09$

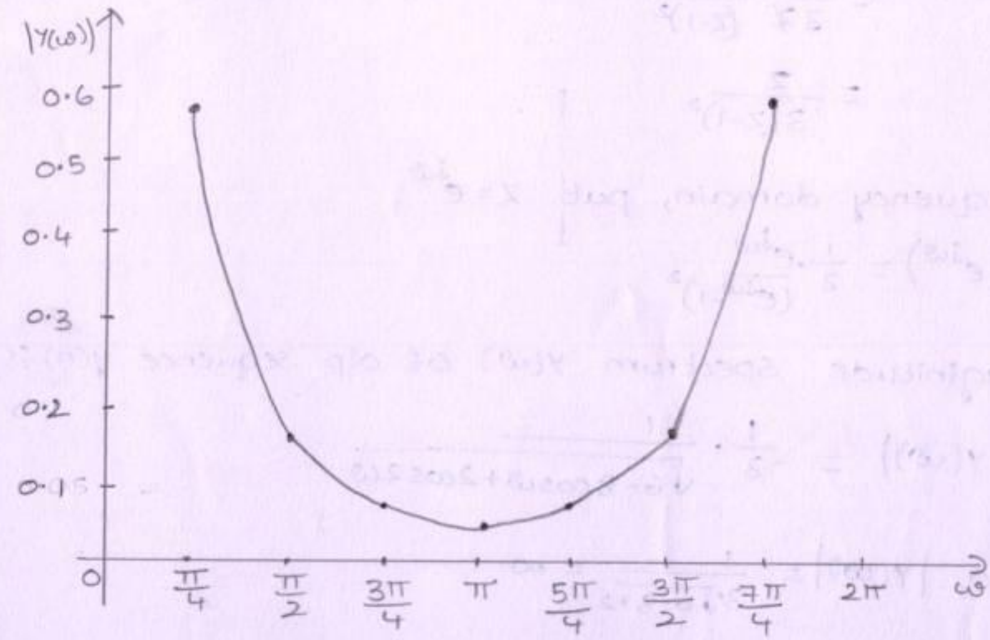
At $\omega = \pi \Rightarrow |Y(\omega)| = \frac{1}{3} \cdot \frac{1}{\sqrt{6-8\cos\pi+2\cos 2(\pi)}} = 0.083$

At $\omega = \frac{5\pi}{4} \Rightarrow |Y(\omega)| = \frac{1}{3} \cdot \frac{1}{\sqrt{6-8\cos\left(\frac{5\pi}{4}\right)+2\cos 2\left(\frac{5\pi}{4}\right)}} = 0.09$

At $\omega = \frac{3\pi}{2} \Rightarrow |Y(\omega)| = \frac{1}{3} \cdot \frac{1}{\sqrt{6-8\cos\left(\frac{3\pi}{2}\right)+2\cos 2\left(\frac{3\pi}{2}\right)}} = 0.16$

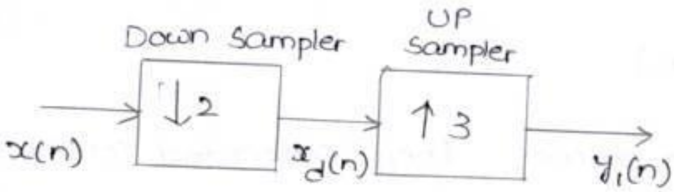
At $\omega = \frac{7\pi}{4} \Rightarrow |Y(\omega)| = \frac{1}{3} \cdot \frac{1}{\sqrt{6-8\cos\left(\frac{7\pi}{4}\right)+2\cos 2\left(\frac{7\pi}{4}\right)}} = 0.56$

At $\omega = 2\pi \Rightarrow |Y(\omega)| = \frac{1}{3} \cdot \frac{1}{\sqrt{6-8\cos 2\pi+2\cos 2(2\pi)}} = \infty$



→ Show that a cascaded down sampler (D) and up sampler (I) is interchangeable only when I & D are co-primes for the following sequence, $x(n) = \{1,3,2,-5,4,-1,2,7,8,9 \ldots\ldots\}$

Sol:



CO-prime ⇒ G.C.D of any two numbers should be '1'.

Let $D=2$ and $I=3$ where, these are co-primed numbers.

2) 3 (1
2
(1)) 2 (2
2
0
co-prime

$x(n) = \{1, 3, 2, -5, 4, -1, 2, 7, 8, 9 \ldots\ldots\}$

Output of down sampler is:

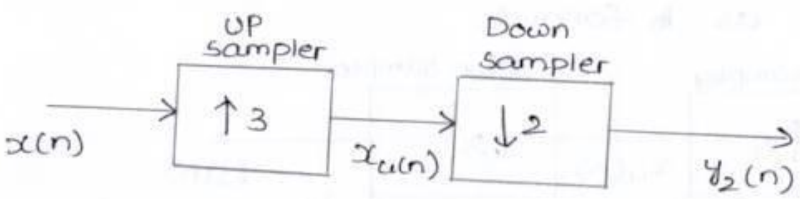
$x_d(n) = \{1, 2, 4, 2, 8, \ldots\ldots\}$ → { Select every Dth sample of i/P sequence

Output of UP sampler is:

$y_1(n) = \{1, 0, 0, 2, 0, 0, 4, 0, 0, 2, 0, 0, 8, \ldots\ldots\}$ — (1)

{ adding $(I-1)$ zero's in b/w samples of i/P sequence

On Interchanging ~~of~~ above cascaded circuit, the resultant circuit obtained is as shown below:



Output of down sampler is:

$x_u(n) = \{1, 0, 0, 3, 0, 0, 2, 0, 0, -5, 0, 0, 4, 0, 0, -1, 0, 0, 2, 0, 0, 7, 0, 0, 8, 0, 0, 9 \ldots\}$

Output of down sampler is:

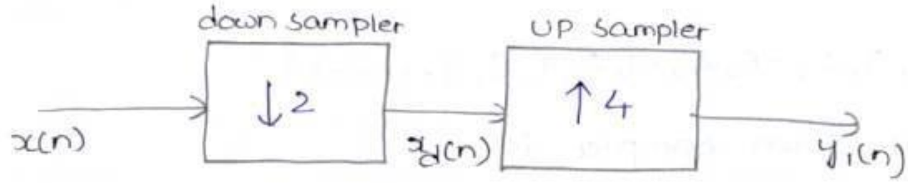
$$y_2(n) = \{1, 0, 0, 2, 0, 0, 4, 0, 0, 2, 0, \ldots\} \quad \text{—} ②$$

From equations ① & ②

$$y_1(n) = y_2(n)$$

If D and I are co-primes, then cascaded connection of down sampler and up sampler can be interchangable.

Case-II:- (Let $D=2$ & $I=4$ i.e., D & I are not co-primes)



$$x(n) = \{1, 3, 2, -5, 4, -1, 2, 7, 8, 9, \ldots\}$$

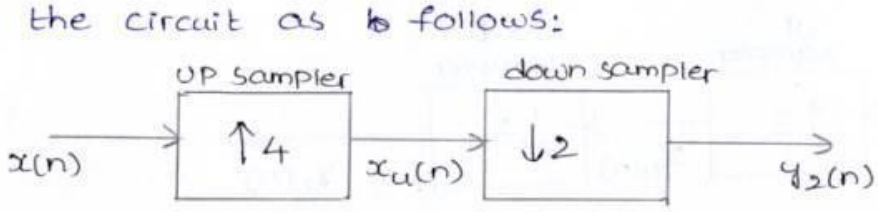
Output of down sampler:

$$x_d(n) = \{1, 2, 4, 2, 8, \ldots\}$$

Output of UP sampler:

$$y_1(n) = \{1, 0, 0, 0, 2, 0, 0, 0, 4, 0, 0, 0, 2, 0, 0, 0, 8, \ldots\} \quad \text{—} ①$$

Interchanging the above cascaded ~~cer~~ blocks, then we have the circuit as ~~b~~ follows:



Output of up sampler:

$$x_u(n) = \{1, 0, 0, 0, 3, 0, 0, 0, 2, 0, 0, 0, -5, 0, 0, 0, 4, 0, 0, 0, -1, 0, 0, 0, 2, 0, 0, 0, 7, 0, 0, 0, 8, 0, 0, 0, 9 \ldots\}$$

Output of down Sampler:

$y_2(n) = \{1, 0, 3, 0, 2, 0, -5, 0, 4, 0, -1, 0, 2, 0, 7, 0, 8, 0, 9 \ldots\}$ — ②

From equations ① & ②, $y_1(n) \neq y_2(n)$

∴ When D & I are not co-primes, then cascaded blocks of down sampler and up sampler are not interchangeable.

→ Sampling rate conversion by polyphase decomposition:

In polyphase decomposition, the n^{th} order filter is decomposed into L sub-sections and they are arranged in parallel.

There are two types of polyphase decomposition:

1. FIR polyphase decomposition.
2. IIR polyphase decomposition.

Polyphase decomposition of FIR filters (or) systems:

Type-I decomposition:

The transfer function of FIR system is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{N-1} z^{-n} h(n)$$

$$= z^{0} h(0) + z^{-1} h(1) + z^{-2} h(2) + z^{-3} h(3) + z^{-4} h(4) + z^{-5} h(5) + z^{-6} h(6) + z^{-7} h(7) + z^{-8} h(8) + z^{-9} h(9) + \cdots \quad \text{—①}$$

(i) Decompose the above equation — ① i.e., H(z) into two sections:

$$\therefore H(z) = \left(z^{0} h(0) + z^{-2} h(2) + z^{-4} h(4) + z^{-6} h(6) + z^{-8} h(8) + \cdots\right) + \left(z^{-1} h(1) + z^{-3} h(3) + z^{-5} h(5) + z^{-7} h(7) + z^{-9} h(9) + \cdots\right)$$

$$\Rightarrow H(z) = \left(z^{0}h(0) + z^{-2}h(2) + z^{-4}h(4) + z^{-6}h(6) + z^{-8}h(8) + \cdots\cdots\right)$$
$$+ z^{-1}\left(z^{0}h(1) + z^{-2}h(3) + z^{-4}h(5) + z^{-6}h(7) + z^{-8}h(9) + \cdots\cdots\right)$$
$$= \left[(z^2)^{0}h(0) + (z^2)^{-1}h(2) + (z^2)^{-2}h(4) + (z^2)^{-3}h(6) + (z^2)^{-4}h(8) + \cdots\cdots\right]$$
$$+ z^{-1}\left[(z^2)^{0}h(1) + (z^2)^{-1}h(3) + (z^2)^{-2}h(5) + (z^2)^{-3}h(7) + (z^2)^{-4}h(9) + \cdots\cdots\right]$$

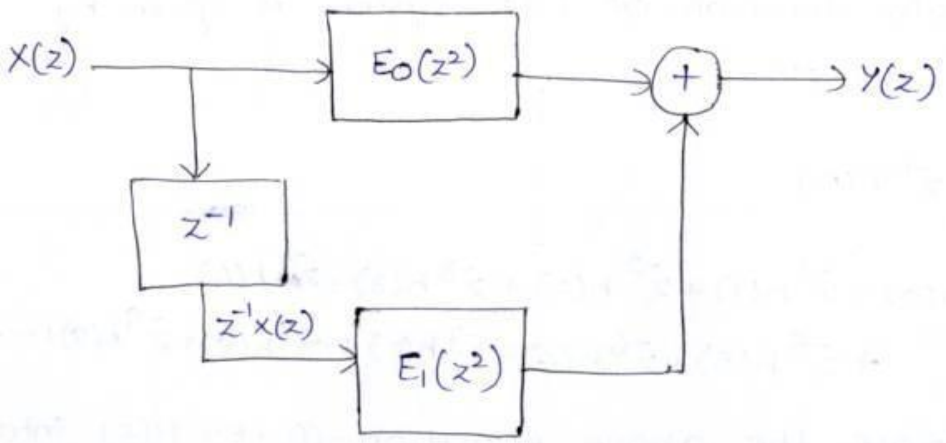
$$\Rightarrow H(z) = E_0(z^2) + z^{-1}\left[E_1(z^2)\right]$$

where,

$$E_0(z^2) = (z^2)^{0}h(0) + (z^2)^{-1}h(2) + (z^2)^{-2}h(4) + (z^2)^{-3}h(6) + (z^2)^{-4}h(8) + \cdots\cdots$$
$$E_1(z^2) = (z^2)^{0}h(1) + (z^2)^{-1}h(3) + (z^2)^{-2}h(5) + (z^2)^{-3}h(7) + (z^2)^{-4}h(9) + \cdots\cdots$$

$$\Rightarrow \frac{Y(z)}{X(z)} = E_0(z^2) + z^{-1}\left[E_1(z^2)\right]$$

$$\Rightarrow Y(z) = \left\{E_0(z^2) + z^{-1}\left[E_1(z^2)\right]\right\} X(z)$$



(ii) Decompose the H(z) [Eq (1)] into three sections:

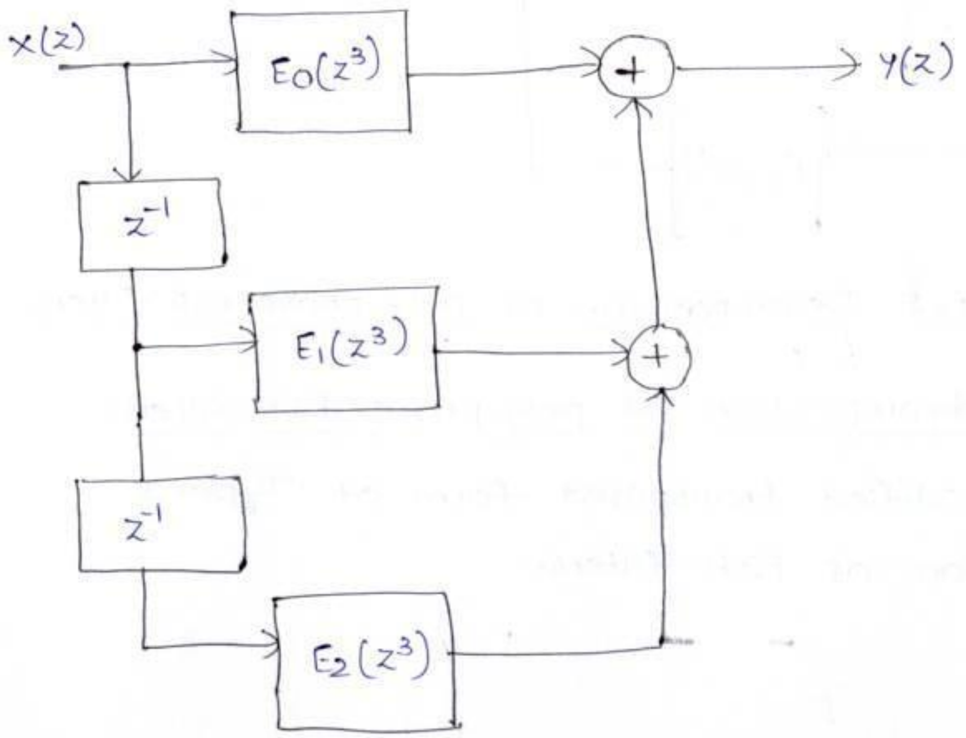
$$H(z) = \left[z^{0}h(0) + z^{-3}h(3) + z^{-6}h(6) + z^{-9}h(9) + \cdots\cdots\right]$$
$$+ \left[z^{-1}h(1) + z^{-4}h(4) + z^{-7}h(7) + z^{-10}h(10) + \cdots\cdots\right]$$
$$+ \left[z^{-2}h(2) + z^{-5}h(5) + z^{-8}h(8) + z^{-11}h(11) + \cdots\cdots\right]$$

$$\Rightarrow H(z) = \left[z^0 h(0) + z^{-3} h(3) + z^{-6} h(6) + z^{-9} h(9) + \cdots\right]$$
$$+ z^{-1}\left[z^0 h(1) + z^{-3} h(4) + z^{-6} h(7) + z^{-9} h(10) + \cdots\right]$$
$$+ z^{-2}\left[z^0 h(2) + z^{-3} h(5) + z^{-6} h(8) + z^{-9} h(11) + \cdots\right]$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \left[(z^3)^0 h(0) + (z^3)^{-1} h(3) + (z^3)^{-2} h(6) + (z^3)^{-3} h(9) + \cdots\right]$$
$$+ z^{-1}\left[(z^3)^0 h(1) + (z^3)^{-1} h(4) + (z^3)^{-2} h(7) + (z^3)^{-3} h(10) + \cdots\right]$$
$$+ z^{-2}\left[(z^3)^0 h(2) + (z^3)^{-1} h(5) + (z^3)^{-2} h(8) + (z^3)^{-3} h(11) + \cdots\right]$$

$$\Rightarrow \frac{Y(z)}{X(z)} = E_0(z^3) + z^{-1}\left[E_1(z^3)\right] + z^{-2}\left[E_2(z^3)\right]$$

$$\Rightarrow Y(z) = \left\{E_0(z^3) + z^{-1}\left[E_1(z^3)\right] + z^{-2}\left[E_2(z^3)\right]\right\} X(z)$$



(iii) Similarly if H(z) is decomposed into 'L' no. of sections then

$$\frac{Y(z)}{X(z)} = E_0(z^L) + z^{-1} E_1(z^L) + z^{-2} E_2(z^L) + z^{-3} E_3(z^L) + \cdots + z^{-(L-1)} \cdot E_{L-1}(z^L)$$

$$\Rightarrow Y(z) = E_0(z^L)\cancel{+z^{-}}\,X(z) + z^{-1}E_1(z^L)X(z) + z^{-2}E_2(z^L)X(z) + \cdots\cdots + z^{-(L-1)}E_{L-1}(z^L)X(z)$$

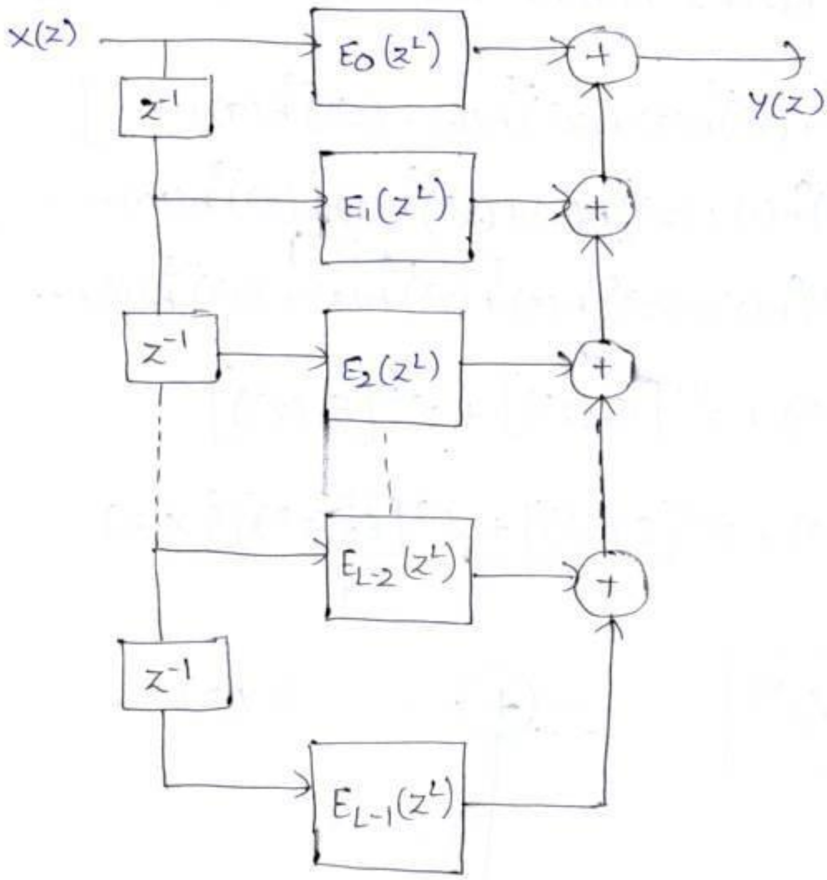
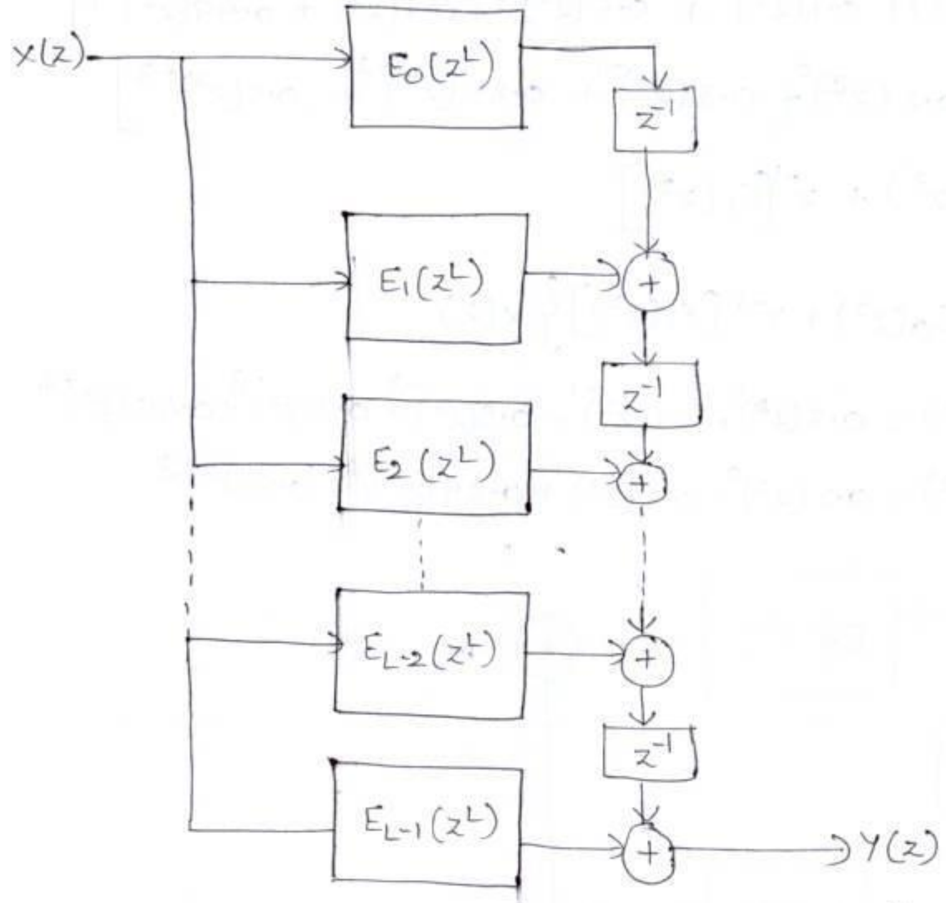


Fig: Type-I Decomposition of poly-phase FIR filters

→ Type-II decomposition of poly-phase FIR filters:

It is modified transposed form of Type-I decomposition of FIR filters.



Problem:

1. The transfer function of FIR filter is

$$H(z) = 0.8 + 0.2z^{-1} + 0.1z^{-2} + 0.3z^{-3} + 0.5z^{-4} + 0.85z^{-5} + 0.7z^{-6} + 0.8z^{-7} + 0.49z^{-8}$$

Perform poly-phase decomposition of H(z) into:

(a) 2 Sections

(b) 3 sections

(c) 4 Sections

Sol: (a) Decompose the given H(z) into two sections:

$$\Rightarrow H(z) = [0.8 + 0.1z^{-2} + 0.5z^{-4} + 0.7z^{-6} + 0.49z^{-8}] + [0.2z^{-1} + 0.3z^{-3} + 0.85z^{-5} + 0.8z^{-7}]$$

$$= [0.8 + 0.1z^{-2} + 0.5z^{-4} + 0.7z^{-6} + 0.49z^{-8}] + z^{-1}[0.2 + 0.3z^{-2} + 0.85z^{-4} + 0.8z^{-6}]$$

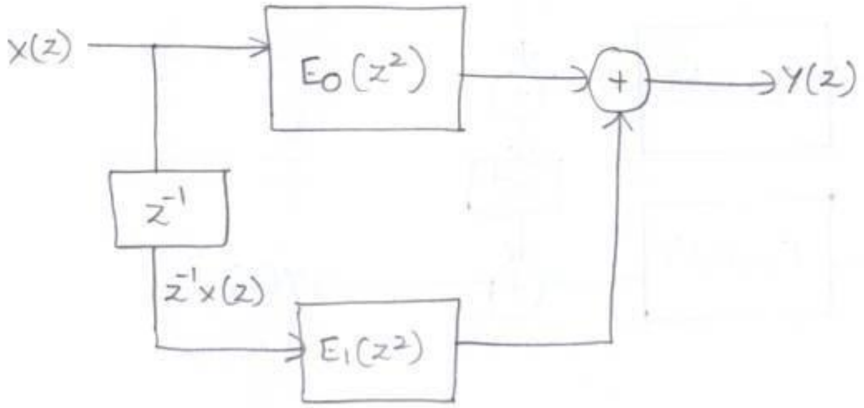
$$\Rightarrow H(z) = \left[0.8(z^2)^0 + 0.1(z^2)^{-1} + 0.5(z^2)^{-2} + 0.7(z^2)^{-3} + 0.49(z^2)^{-4}\right]$$
$$+ z^{-1}\left[0.2(z^2)^0 + 0.3(z^2)^{-1} + 0.85(z^2)^{-2} + 0.8(z^2)^{-3}\right]$$

$$\Rightarrow \frac{Y(z)}{X(z)} = E_0(z^2) + z^{-1}\left[E_1(z^2)\right]$$

$$\Rightarrow Y(z) = \left\{E_0(z^2) + z^{-1}\left[E_1(z^2)\right]\right\} X(z)$$

where, $E_0(z^2) = 0.8(z^2)^0 + 0.1(z^2)^{-1} + 0.5(z^2)^{-2} + 0.7(z^2)^{-3} + 0.49(z^2)^{-4}$

$E_1(z^2) = 0.2(z^2)^0 + 0.3(z^2)^{-1} + 0.85(z^2)^{-2} + 0.8(z^2)^{-3}$



(b) Decomposition of H(z) into 3 Sections:

WKT,

$$H(z) = 0.8 + 0.2z^{-1} + 0.1z^{-2} + 0.3z^{-3} + 0.5z^{-4} + 0.85z^{-5} + 0.7z^{-6} + 0.8z^{-7} + 0.49z^{-8}$$

On decomposing the above H(z) equation into 3 sections we have

$$H(z) = (0.8 + 0.3z^{-3} + 0.7z^{-6}) + (0.2z^{-1} + 0.5z^{-4} + 0.8z^{-7})$$
$$+ (0.1z^{-2} + 0.85z^{-5} + 0.49z^{-8})$$
$$= (0.8 + 0.3z^{-3} + 0.7z^{-6}) + z^{-1}(0.2 + 0.5z^{-3} + 0.8z^{-6})$$
$$+ z^{-2}(0.1 + 0.85z^{-3} + 0.49z^{-6})$$

$$\Rightarrow H(z) = \left[0.8(z^3)^0 + 0.3(z^3)^{-1} + 0.7(z^3)^{-2}\right] + z^{-1}\left[0.2(z^3)^0 + 0.5(z^3)^{-1} + 0.8(z^3)^{-2}\right]$$
$$+ z^{-2}\left[0.1(z^3)^0 + 0.85(z^3)^{-1} + 0.49(z^3)^{-2}\right]$$

$$= E_0(z^3) + z^{-1}E_1(z^3) + z^{-2}E_2(z^3)$$

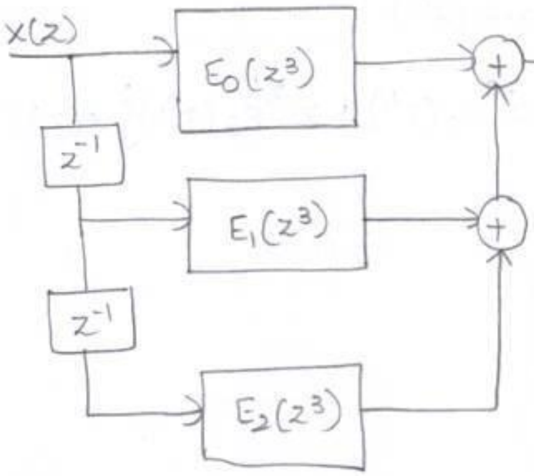
where, $E_0(z^3) = 0.8(z^3)^0 + 0.3(z^3)^{-1} + 0.7(z^3)^{-2}$

$E_1(z^3) = 0.2(z^3)^0 + 0.5(z^3)^{-1} + 0.8(z^3)^{-2}$

$E_2(z^3) = 0.1(z^3)^0 + 0.85(z^3)^{-1} + 0.49(z^3)^{-2}$

$$\Rightarrow \frac{Y(z)}{X(z)} = E_0(z^3) + z^{-1}E_1(z^3) + z^{-2}E_2(z^3)$$

$$\Rightarrow Y(z) = \left\{E_0(z^3) + z^{-1}E_1(z^3) + z^{-2}E_2(z^3)\right\}X(z)$$



(c) Decomposition of H(z) into 4-Sections:

WKT, $H(z) = 0.8 + 0.2z^{-1} + 0.1z^{-2} + 0.3z^{-3} + 0.5z^{-4} + 0.85z^{-5} + 0.7z^{-6} + 0.8z^{-7} + 0.49z^{-8}$

On decomposing the above H(z) equation into 4-Sections, we have

$$H(z) = \left(0.8 + 0.5z^{-4} + 0.49z^{-8}\right) + \left(0.2z^{-1} + 0.85z^{-5}\right)$$
$$+ \left(0.1z^{-2} + 0.7z^{-6}\right) + \left(0.3z^{-3} + 0.8z^{-7}\right)$$

$$H(z) = (0.8 + 0.5z^{-4} + 0.49z^{-8}) + z^{-1}(0.2 + 0.85z^{-4})$$
$$+ z^{-2}(0.1 + 0.7z^{-4}) + z^{-3}(0.3 + 0.8z^{-4})$$
$$= \left(0.8(z^4)^0 + 0.5(z^4)^{-1} + 0.49(z^4)^{-2}\right) + z^{-1}\left(0.2(z^4)^0 + 0.85(z^4)^{-1}\right)$$
$$+ z^{-2}\left(0.1(z^4)^0 + 0.7(z^4)^{-1}\right) + z^{-3}\left(0.3(z^4)^0 + 0.8(z^4)^{-1}\right)$$

$$\frac{Y(z)}{X(z)} = E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4)$$

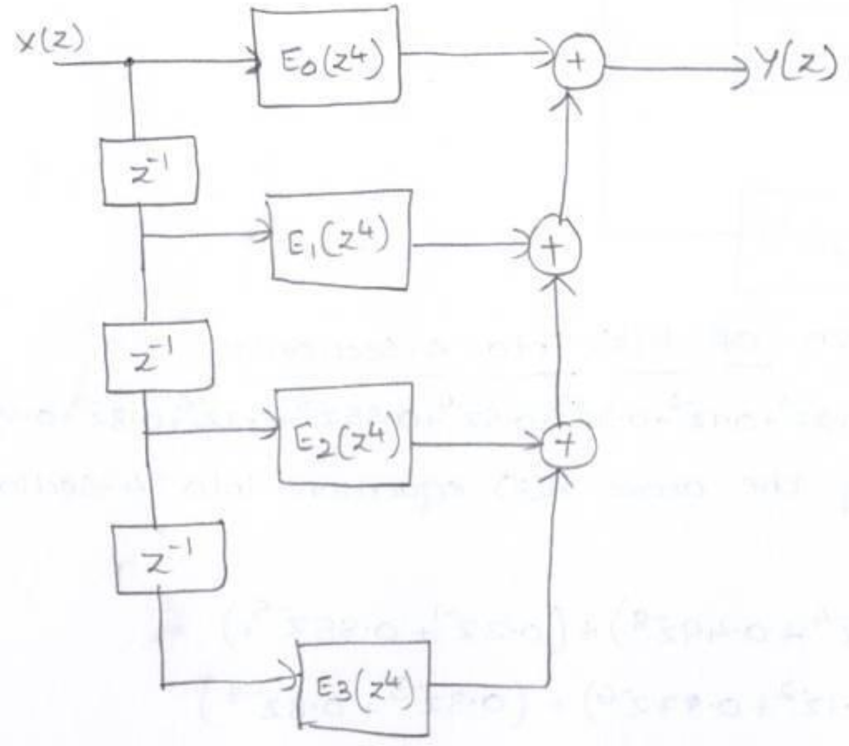
where, $E_0(z^4) = 0.8(z^4)^0 + 0.5(z^4)^{-1} + 0.49(z^4)^{-2}$

$E_1(z^4) = 0.2(z^4)^0 + 0.85(z^4)^{-1}$

$E_2(z^4) = 0.1(z^4)^0 + 0.7(z^4)^{-1}$

$E_3(z^4) = 0.3(z^4)^0 + 0.8(z^4)^{-1}$

$$\Rightarrow Y(z) = \left\{E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4)\right\}X(z)$$



→ The transfer function of IIR filter $H(z) = \frac{1+0.9z^{-1}}{1-0.7z^{-1}}$.

Perform polyphase decomposition of $H(z)$ into

(a) 2 sections

(b) 4 sections

Sol:

Note:-

The transfer function of IIR filter:

$$H(z) = \frac{\sum_{m=0}^{N} a_m z^{-m}}{1 + \sum_{m=1}^{N} b_m z^{-m}}$$

(a) Decomposition of $H(z)$ into 2 sections:

Standard $H(z)$ form for 2 sections is: $H(z) = E_0(z^2) + z^{-1}E_1(z^2)$

$$H(z) = \frac{1+0.9z^{-1}}{1-0.7z^{-1}} \times \frac{1+0.7z^{-1}}{1+0.7z^{-1}}$$

$$= \frac{1+0.9z^{-1}+0.7z^{-1}+0.63z^{-2}}{1-0.49z^{-2}}$$

$$= \frac{1+z^{-1}(1.6)+0.63z^{-2}}{1-0.49z^{-2}}$$

$$= \frac{(1+0.63z^{-2})+1.6z^{-1}}{1-0.49z^{-2}}$$

$$= \frac{1+0.63z^{-2}}{1-0.49z^{-2}} + \frac{1.6z^{-1}}{1-0.49z^{-2}}$$

$$= \frac{1+0.63z^{-2}}{1-0.49z^{-2}} + z^{-1}\left[\frac{1.6}{1-0.49z^{-2}}\right] \quad \text{—— (i)}$$

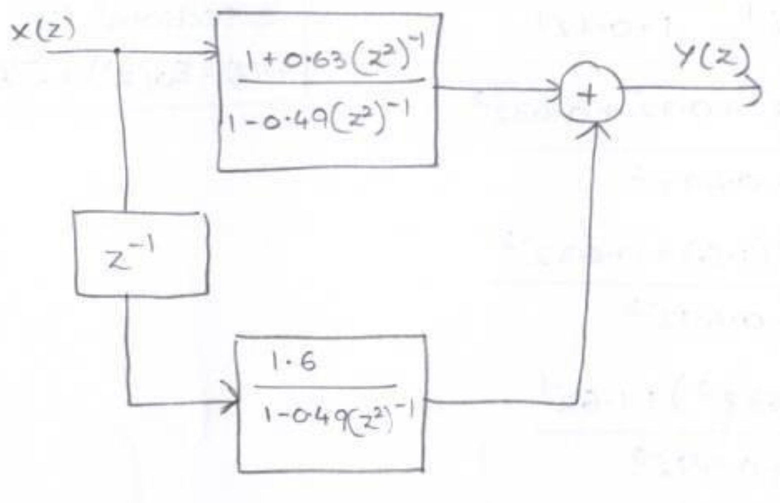
$$= \frac{1+0.63(z^2)^{-1}}{1-0.49(z^2)^{-1}} + z^{-1}\left[\frac{1.6}{1-0.49(z^2)^{-1}}\right] \quad \text{—— (1)}$$

WKT, standard form for Polyphase decomposition for two sections is: $H(z) = E_0(z^2) + z^{-1}E_1(z^2)$ —— (2)

On comparing equation ① and equation ②, we can say that equation ① represents H(z) for polyphase decomposed form of two sections

$$\frac{Y(z)}{X(z)} = \frac{1+0.63(z^2)^{-1}}{1-0.49(z^2)^{-1}} + z^{-1}\left[\frac{1.6}{1-0.49(z^2)^{-1}}\right]$$

$$\Rightarrow Y(z) = \Bigg\{\underbrace{\frac{1+0.63(z^2)^{-1}}{1-0.49(z^2)^{-1}}}_{E_0(z^2)} + z^{-1}\underbrace{\left(\frac{1.6}{1-0.49(z^2)^{-1}}\right)}_{E_1(z^2)}\Bigg\} X(z)$$



(b) Decomposition of H(z) into 4 sections:

$$H(z) = \frac{1+0.9z^{-1}}{1-0.7z^{-1}} \times \frac{1+0.7z^{-1}}{1+0.7z^{-1}}$$

$$= \frac{1+0.9z^{-1}+0.7z^{-1}+0.63z^{-2}}{1-0.49z^{-2}}$$

$$= \frac{1+z^{-1}(1.6)+0.63z^{-2}}{1-0.49z^{-2}}$$

$$\Rightarrow H(z) = \frac{1+1.6z^{-1}+0.63z^{-2}}{1-0.49z^{-2}} \times \frac{1+0.49z^{-2}}{1+0.49z^{-2}}$$

$$= \frac{1+1.6z^{-1}+0.63z^{-2}+0.49z^{-2}+0.78z^{-3}+0.3087z^{-4}}{1-0.2401z^{-4}}$$

$$= \frac{1+1.6z^{-1}+1.12z^{-2}+0.784z^{-3}+0.3087z^{-4}}{1-0.2401z^{-4}}$$

$$= \frac{(1+0.38z^{-4})+z^{-1}(1.6)+z^{-2}(1.12)+z^{-3}(0.784)}{1-0.2401z^{-4}}$$

$$= \frac{1+0.38z^{-4}}{1-0.24z^{-4}} + \frac{z^{-1}(1.6)}{1-0.2401z^{-4}} + \frac{z^{-2}(1.12)}{1-0.2401z^{-4}} + \frac{z^{-3}(0.784)}{1-0.2401z^{-4}}$$

$$= \frac{1+0.38(z^4)^{-1}}{1-0.24(z^4)^{-1}} + z^{-1}\frac{(1.6)}{1-0.2401(z^4)^{-1}} + z^{-2}\frac{(1.12)}{1-0.2401(z^4)^{-1}} + z^{-3}\left(\frac{0.784}{1-0.2401(z^4)^{-1}}\right) \quad \text{— ①}$$

Standard form of polyphase decomposition of $H(z)$ into 4 sections is given by:

$$H(z) = E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4) \quad \text{— ②}$$

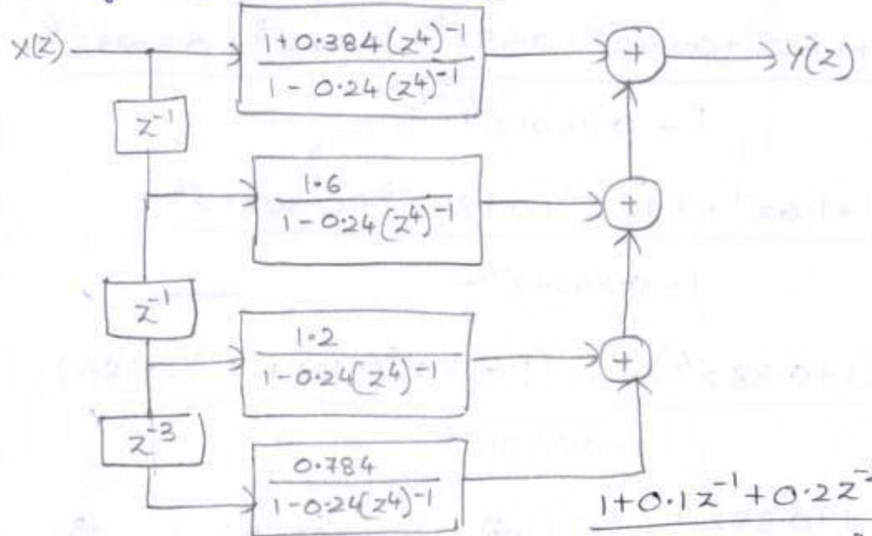
On comparing equations ① & ②, equation-① is said to be a polyphase decomposed $H(z)$ into 4 sections.

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1+0.38(z^4)^{-1}}{1-0.24(z^4)^{-1}} + z^{-1}\left[\frac{1.6}{1-0.24(z^4)^{-1}}\right] + z^{-2}\left[\frac{1.12}{1-0.2401(z^4)^{-1}}\right] + z^{-3}\left[\frac{0.784}{1-0.2401(z^4)^{-1}}\right]$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \left\{\frac{1+0.38(z^4)^{-1}}{1-0.24(z^4)^{-1}} + z^{-1}\left[\frac{1.6}{1-0.24(z^4)^{-1}}\right] + z^{-2}\left[\frac{1.12}{1-0.2401(z^4)^{-1}}\right] + z^{-3}\left[\frac{0.784}{1-0.24(z^4)^{-1}}\right]\right\} \quad \text{— (3)}$$

Standard form of polyphase decomposition of H(z) is given by, $H(z) = E_0(z^4) + E_1(z^4)z^{-1} + E_2 z^{-2}(z^4) + E_3 z^{-3}(z^4)$ — ②

comparing equations ① & ②



→ The transfer function of $H(z) = \dfrac{1+0.1z^{-1}+0.2z^{-2}}{1-0.2z^{-1}+0.3z^{-2}}$

Perform polyphase decomposition of H(z) into:

(a) 2 sections

(b) 4 sections

sol: (a) Decomposition of H(z) into 2 sections:

Standard H(z) equation for 2 sections is:

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

$$H(-z) = E_0(z^2) - z^{-1}E_1(z^2)$$

$$\therefore \quad E_0(z^2) = \frac{1}{2}\left[H(z) + H(-z)\right]$$

$$z^{-1}E_1(z^2) = \frac{1}{2}\left[H(z) - H(-z)\right]$$

$$E_0(z^2) = \frac{1}{2}\left[\frac{1+0.1z^{-1}+0.2z^{-2}}{1-0.2z^{-1}+0.3z^{-2}} + \frac{1-0.1z^{-1}+0.2z^{-2}}{1+0.2z^{-1}+0.3z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{(1+0.1z^{-1}+0.2z^{-2})(1+0.2z^{-1}+0.3z^{-2}) + (1-0.2z^{-1}+0.3z^{-2})(1-0.1z^{-1}+0.2z^{-2})}{(1-0.2z^{-1}+0.3z^{-2})(1+0.2z^{-1}+0.3z^{-2})}\right]$$

$$E_0(z^2) = \frac{1}{2}\left[\frac{\begin{aligned}&1+0.2z^{-1}+0.3z^{-2}+0.1z^{-1}+0.02z^{-2}+0.03z^{-3}+0.2z^{-2}+0.04z^{-3}+0.06z^{-4}\\ &+1-0.1z^{-1}+0.2z^{-2}+0.02z^{-2}-0.2z^{-1}-0.04z^{-3}+0.3z^{-2}-0.03z^{-3}+0.06z^{-4}\end{aligned}}{1+0.2z^{-1}+0.3z^{-2}-0.2z^{-1}-0.04z^{-2}-0.06z^{-3}+0.3z^{-2}+0.06z^{-3}+0.09z^{-4}}\right]$$

$$= \frac{1}{2}\left[\frac{2+1.04z^{-2}+0.12z^{-4}}{1+0.6z^{-2}-0.04z^{-2}+0.09z^{-4}}\right]$$

$$= \frac{1}{2}\left[\frac{2+1.04z^{-2}+0.12z^{-4}}{1+0.56z^{-2}+0.09z^{-4}}\right]$$

$$z^{-1}E_1(z^2) = \frac{1}{2}\left[H(z)-H(-z)\right]$$

$$= \frac{1}{2}\left[\frac{1+0.1z^{-1}+0.2z^{-2}}{1-0.2z^{-1}+0.3z^{-2}} - \frac{1-0.1z^{-1}+0.2z^{-2}}{1+0.2z^{-1}+0.3z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{(1+0.1z^{-1}+0.2z^{-2})(1+0.2z^{-1}+0.3z^{-2}) - (1-0.1z^{-1}+0.2z^{-2})(1-0.2z^{-1}+0.3z^{-2})}{(1-0.2z^{-1}+0.3z^{-2})(1+0.2z^{-1}+0.3z^{-2})}\right]$$

$$= \frac{1}{2}\left[\frac{\begin{aligned}&1+0.2z^{-1}+0.3z^{-2}+0.1z^{-1}+0.02z^{-2}+0.03z^{-3}+0.2z^{-2}+0.04z^{-3}+0.06z^{-4}\\ &-[1-0.1z^{-1}+0.2z^{-2}-0.2z^{-1}+0.02z^{-2}-0.04z^{-3}+0.3z^{-2}-0.03z^{-3}+0.06z^{-2}]\end{aligned}}{1+0.56z^{-2}+0.09z^{-4}}\right]$$

$$= \frac{1}{2}\left[\frac{\begin{aligned}&0.2z^{-1}+0.3z^{-2}+0.1z^{-1}+0.02z^{-2}+0.03z^{-3}+0.2z^{-2}+0.04z^{-3}+0.06z^{-4}\\ &+0.1z^{-1}-0.2z^{-2}+0.2z^{-1}-0.02z^{-2}+0.04z^{-3}-0.3z^{-2}+0.03z^{-3}-0.06z^{-4}\end{aligned}}{1+0.56z^{-2}+0.09z^{-4}}\right]$$

$$= \frac{1}{2}\left[\frac{0.6z^{-1}-0.06z^{-2}+0.14z^{-3}+0.06z^{-4}}{1+0.56z^{-2}+0.09z^{-4}}\right]$$

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

$$\cancel{E_0(z^2)}\ H(z) = \frac{1}{2}\left[\frac{2+1.04z^{-2}+0.12z^{-4}}{1+0.56z^{-2}+0.09z^{-4}}\right] + \frac{1}{2}z^{-1}\left[\frac{0.6-0.06z^{-1}+0.14z^{-2}+0.06z^{-3}}{1+0.56z^{-2}+0.09z^{-4}}\right]$$

$$= \frac{1}{2}\left[\frac{2+1.04z^{-2}+0.12z^{-4}}{1+0.56z^{-2}+0.09z^{-4}}\right] + z^{-1}\,\frac{1}{2}\left[\frac{0.6-0.06z^{-1}+0.14z^{-2}+0.06z^{-3}}{1+0.56z^{-2}+0.09z^{-4}}\right]$$

$$\cancel{H(z) = \frac{1}{2}\left[\frac{2(z^{-2})^{0}+1.04(z^{-2})^{+2}+0.12(z^{-2})^{+2}}{1(z^{2})^{0}+0.56(z^{-2})^{2}+0.09(z^{-2})^{2}}\right] + z^{-1}\frac{1}{2}\left[\frac{0.6^{2}-0.06(z^{-1})^{2}+0.14(z^{-1})^{1}+0.06(z^{-3/2})^{2}}{1+0.56(z^{-1})^{2}+0.09(z^{-2})^{2}}\right]}$$

$$= E_0(z^2) + z^{-1}E_1(z^2)$$

$$H(z) = \frac{1}{2}\left[\frac{2(z^{0})^{2}+1.04(z^{-1})^{2}+0.12(z^{-2})^{2}}{1+0.56(z^{-1})^{2}+0.09(z^{-2})^{2}}\right] + z^{-1}\frac{1}{2}\left[\frac{0.6(z^{0})^{2}-0.06(z^{-1/2})^{2}+0.14(z^{-1})^{2}+0.06(z^{-3/2})^{2}}{1+0.56(z^{-1})^{2}+0.09(z^{-2})^{2}}\right]$$

(b) Decomposition of H(z) into 4 sections:

$$H(z) = \frac{1}{2}\left[\frac{2+1.04z^{-2}+0.12z^{-4}}{1+0.56z^{-2}+0.09z^{-4}}\right] + \frac{1}{2}\left[\frac{0.6z^{-1}-0.06z^{-2}+0.14z^{-3}+0.06z^{-4}}{1+0.56z^{-2}+0.09z^{-4}}\right]$$

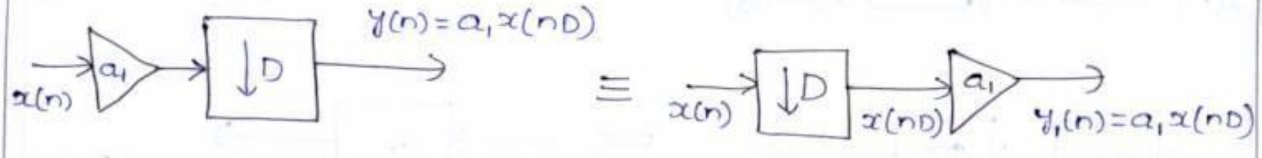
$$= \frac{1}{2\left[1+0.56z^{-2}+0.09z^{-4}\right]}\left[2+1.04z^{-2}+0.12z^{-4}+0.6z^{-1}-0.06z^{-2}+0.14z^{-3}+0.06z^{-4}\right]$$

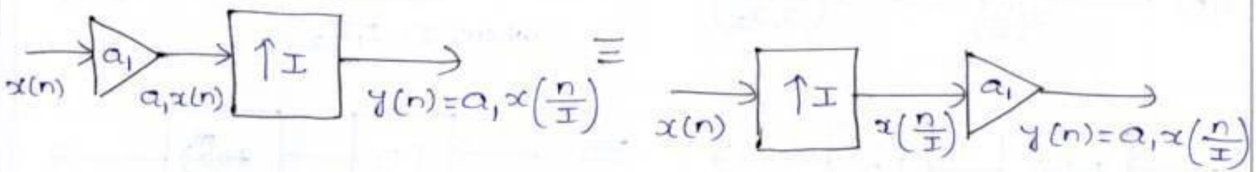
$$= \frac{2+0.6z^{-1}+0.98z^{-2}+0.14z^{-3}+0.18z^{-4}}{2\left[1+0.56z^{-2}+0.09z^{-4}\right]}$$

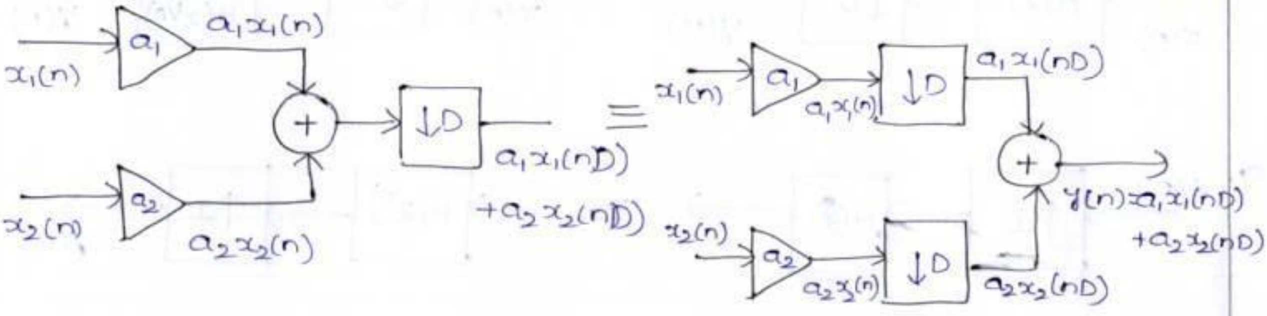
$$= \frac{1}{1+0.56z^{-2}+0.09z^{-4}} + z^{-1}\frac{0.3}{1+0.56z^{-2}+0.09z^{-4}} + z^{-2}\frac{0.49}{1+0.56z^{-2}+0.09z^{-4}} + z^{-3}\frac{0.7}{1+0.56z^{-2}+0.09z^{-4}} + z^{-4}\frac{0.9}{1+0.56z^{-2}+0.09z^{-4}}$$

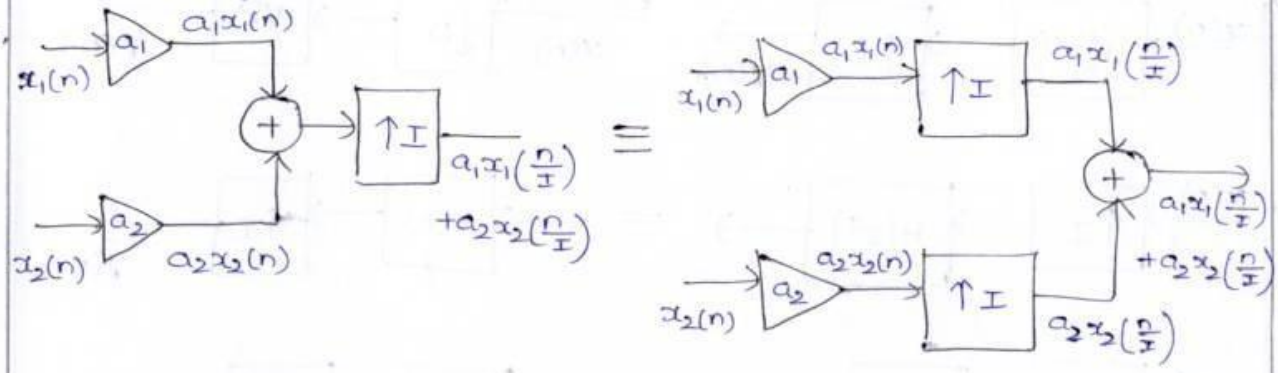
$$= \frac{1}{1+0.56(z^{-2/4})^{4}+0.09(z^{-1})^{4}} + z^{-1}\frac{0.3}{1+0.56(z^{-2/4})^{4}+0.09(z^{-1})^{4}} + z^{-2}\frac{0.49}{1+0.56(z^{-2/4})^{4}+0.09(z^{-1})^{4}} + z^{-3}\frac{0.7}{1+0.56(z^{-2/4})^{4}+0.09(z^{-1})^{4}} + z^{-4}\frac{0.9}{1+0.56(z^{-2/4})^{4}+0.09(z^{-1})^{4}}$$

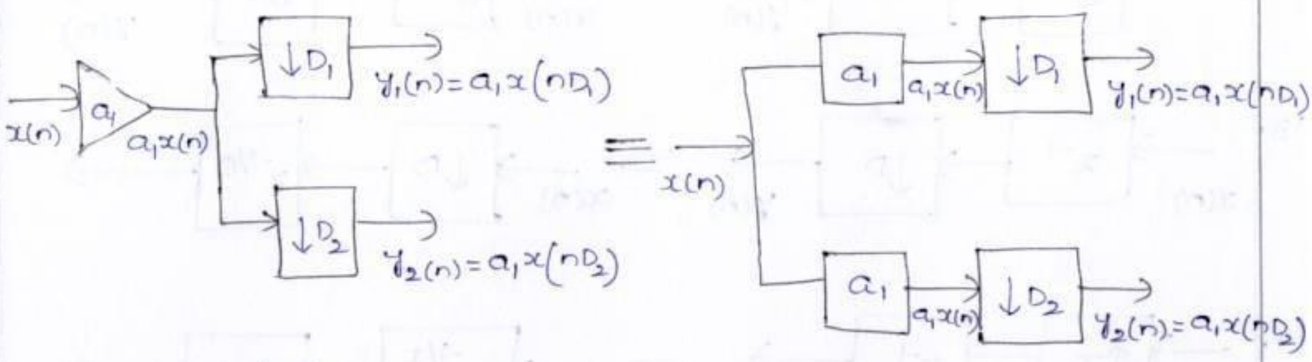
Identities of Multi-rate Digital signal processing:

1. 

2. 

3. 

4. 

5. 

6. $x(n) \rightarrow \boxed{\downarrow D_1} \xrightarrow{x(nD_1)} \boxed{\downarrow D_2} \rightarrow x(nD_1D_2) \quad \equiv \quad x(n) \rightarrow \boxed{D} \rightarrow y(n) = x(nD_1D_2)$

$D = D_1 D_2$

7. $x(n) \rightarrow \boxed{\uparrow I_1} \xrightarrow{x\left(\frac{n}{I_1}\right)} \boxed{\uparrow I_2} \rightarrow x\left(\frac{n}{I_1 I_2}\right) \quad \equiv \quad x(n) \rightarrow \boxed{I} \rightarrow y(n) = x\left(\frac{n}{I}\right)$

where, $I = I_1 I_2$

8. $x(n) \rightarrow \boxed{H(z)} \rightarrow \boxed{\downarrow D} \rightarrow y(n) \quad \equiv \quad x(n) \rightarrow \boxed{\downarrow D} \rightarrow \boxed{\cancel{H(z^D)}\ H(z^{1/D})} \rightarrow y(n)$

9. $x(n) \rightarrow \boxed{\uparrow I} \rightarrow \boxed{H(z)} \rightarrow \quad \equiv \quad \rightarrow \boxed{H(z^I)} \rightarrow \boxed{\uparrow I} \rightarrow y(n)$

10. $x(n) \rightarrow \boxed{H(z^D)} \rightarrow \boxed{\downarrow D} \rightarrow y(n) \quad \equiv \quad x(n) \rightarrow \boxed{\downarrow D} \rightarrow \boxed{H(z)} \rightarrow y(n)$

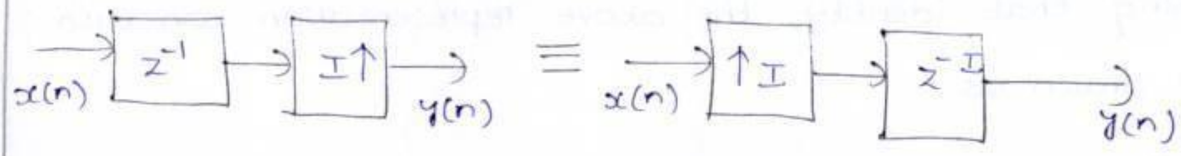
11. $x(n) \rightarrow \boxed{\uparrow I} \rightarrow \boxed{H(z^I)} \rightarrow \quad \equiv \quad \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow I} \rightarrow y(n)$

12. $x(n) \rightarrow \boxed{\downarrow D} \rightarrow \boxed{z^{-1}} \rightarrow y(n) \quad \equiv \quad x(n) \rightarrow \boxed{z^{-D}} \rightarrow \boxed{\downarrow D} \rightarrow y(n)$

13. $x(n) \rightarrow \boxed{z^{-1}} \rightarrow \boxed{\downarrow D} \rightarrow y(n) \quad \equiv \quad x(n) \rightarrow \boxed{\downarrow D} \rightarrow \boxed{z^{-1/D}} \rightarrow$

14. $x(n) \rightarrow \boxed{\uparrow I} \rightarrow \boxed{z^{-1}} \rightarrow y(n) \quad \equiv \quad x(n) \rightarrow \boxed{z^{-1/I}} \rightarrow \boxed{\uparrow I} \rightarrow y(n)$

15.



→ Polyphase decomposition of decimator:

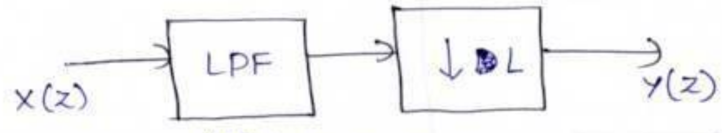
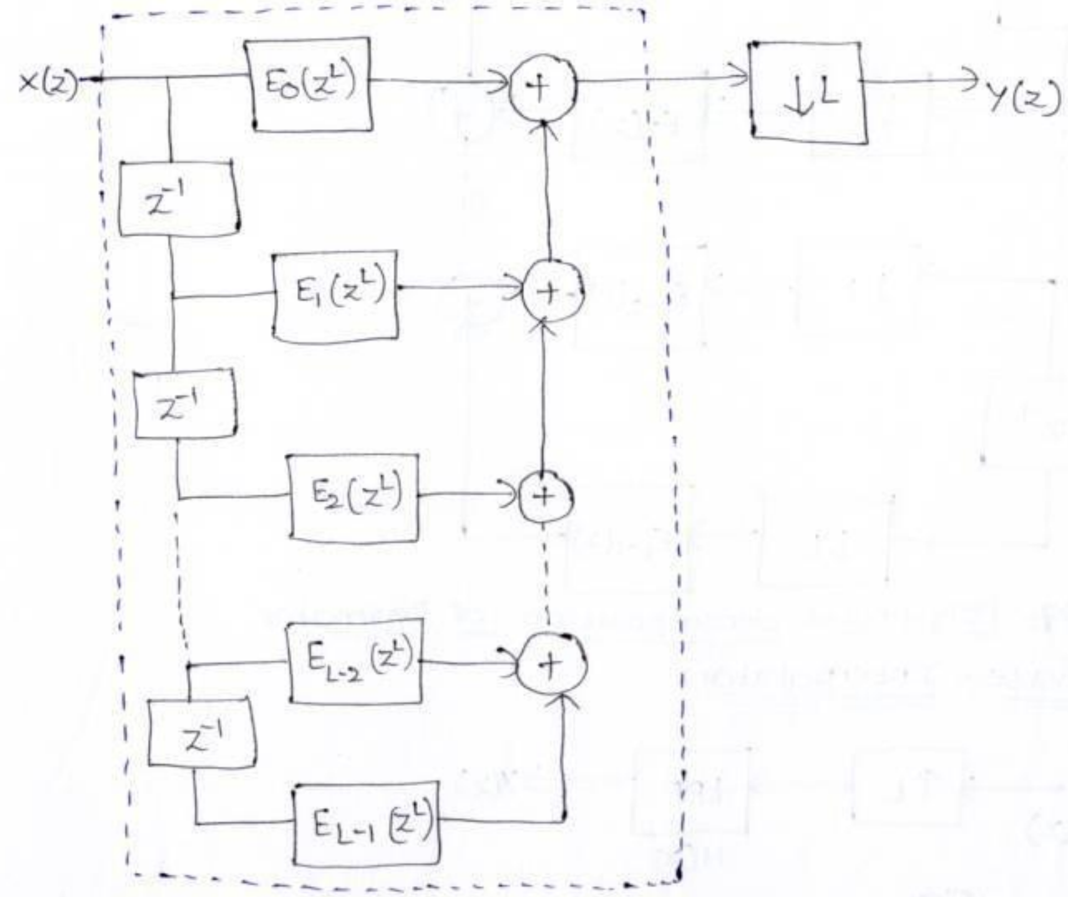
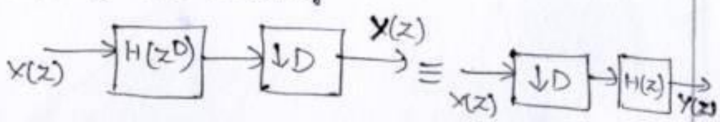


Fig: Decimator

Let $H(z)$ is decomposed into 'L' sub-sections, then poly-phase decomposition of $H(z)$ is shown below.



∵ WKT, the identity



Using that identity, the above representation can also be drawn as

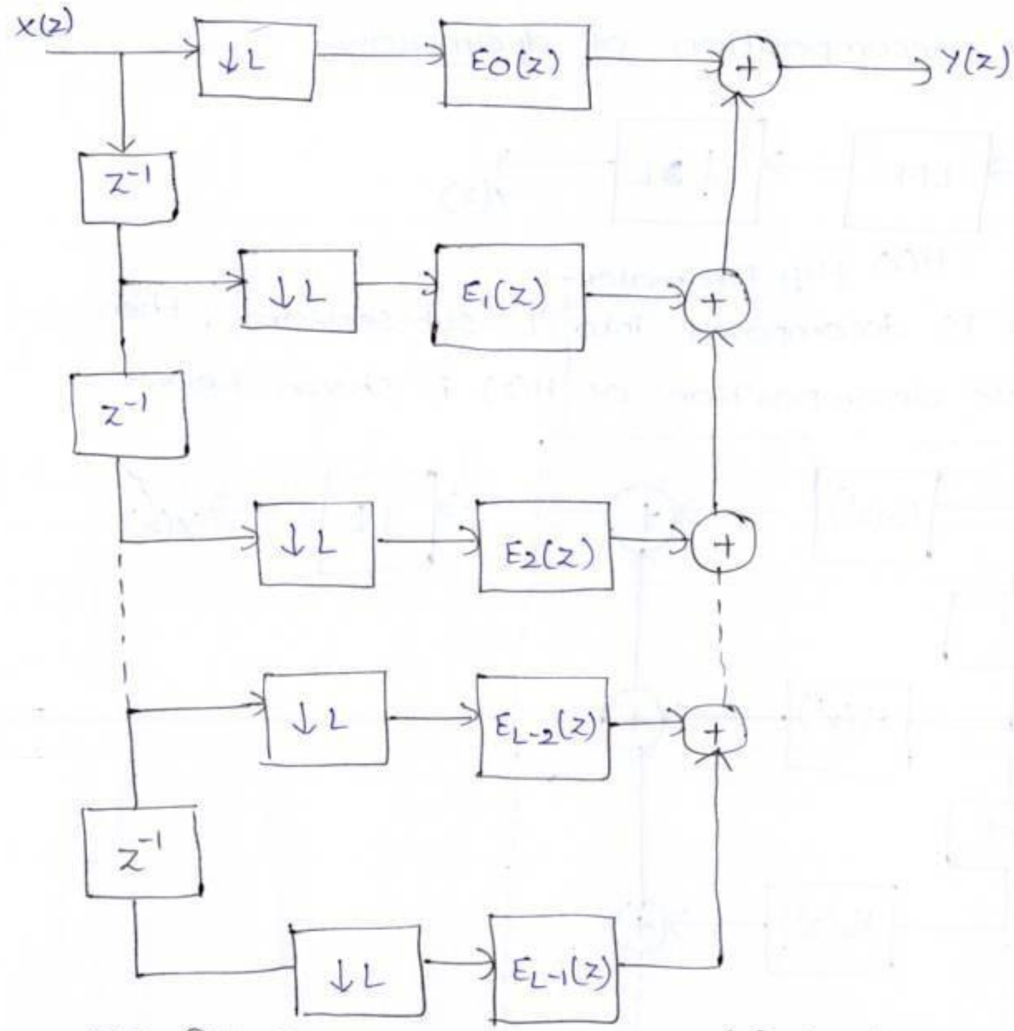


Fig: Poly-Phase decomposition of Decimator

⇒ Polyphase - Interpolator:

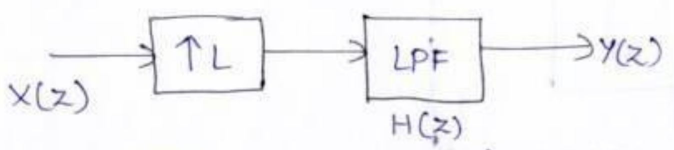
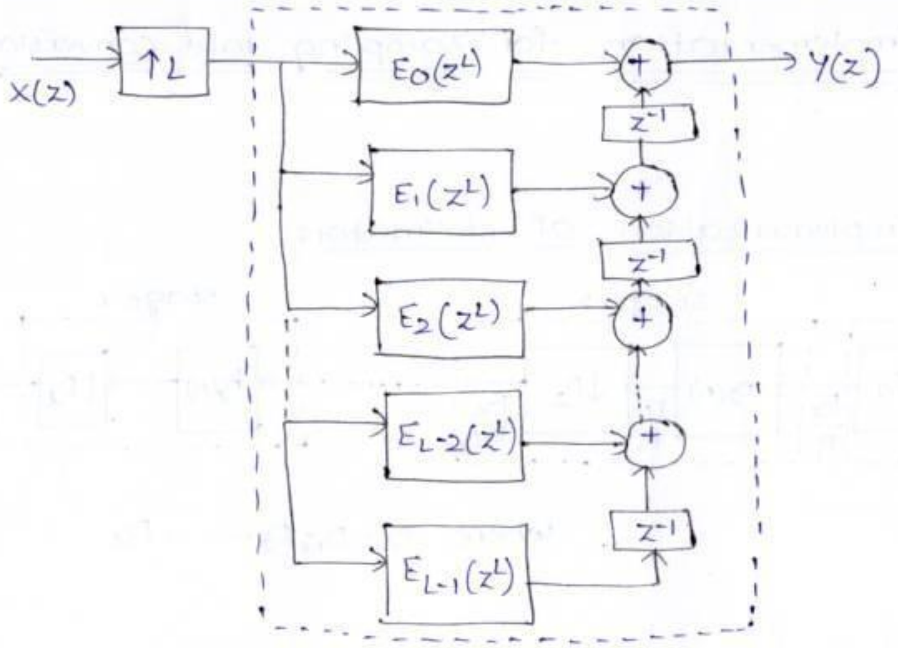
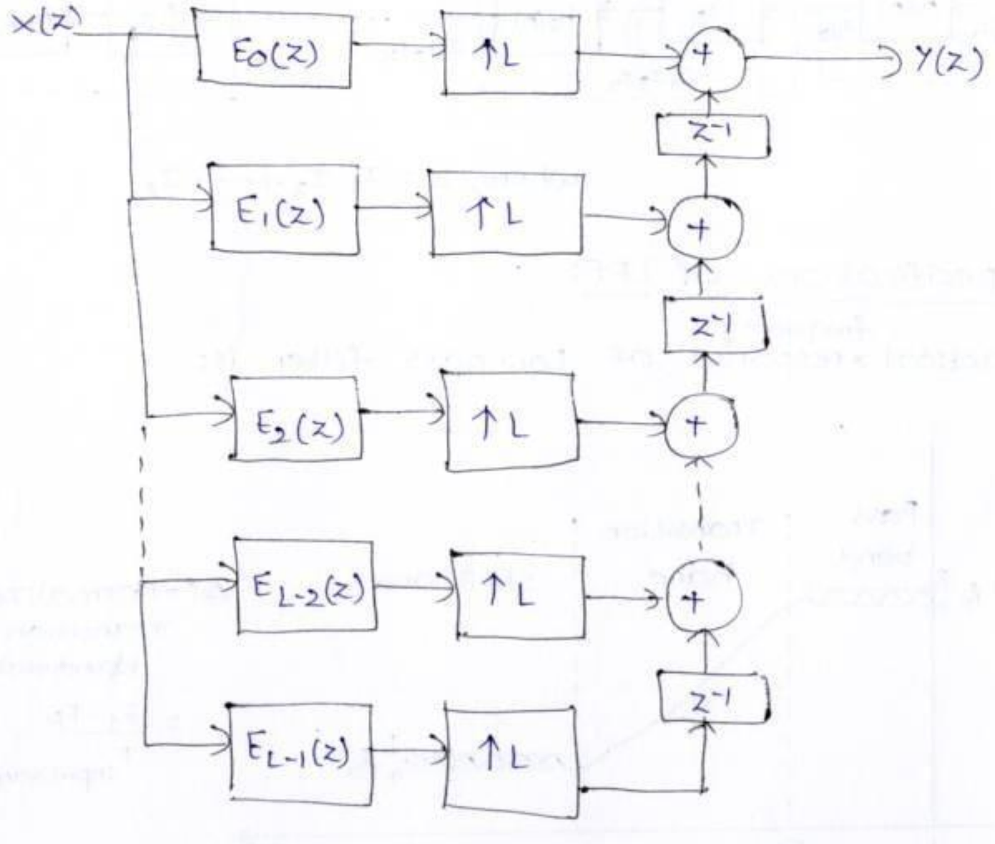


Fig: Interpolator



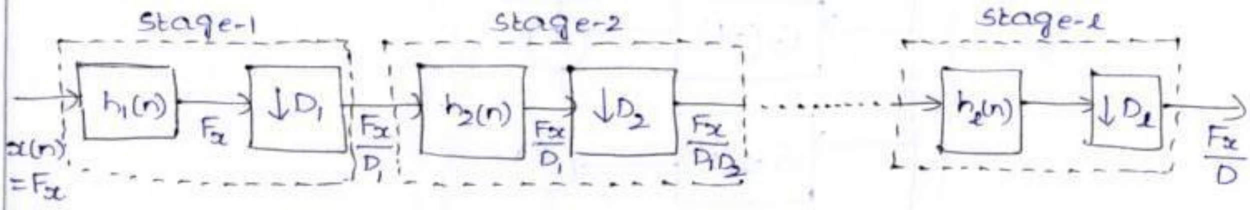
Using the one of the identities, the above representation can also be drawn as shown in below figure:



⇒ Multi-stage implementation for sampling rate conversion:

Case-1:-

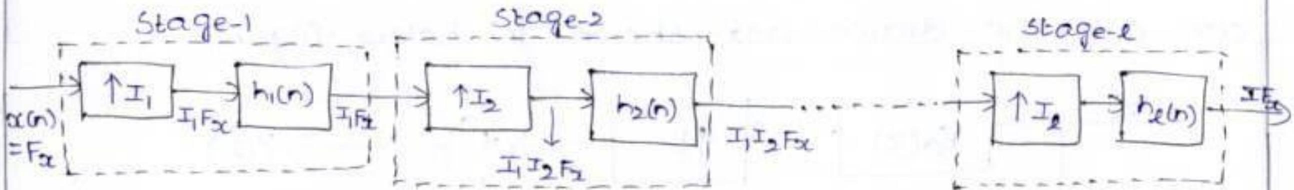
Multi-Stage implementation of decimator:



where, $D = D_1 \cdot D_2 \cdots\cdots D_l$

Case-2:-

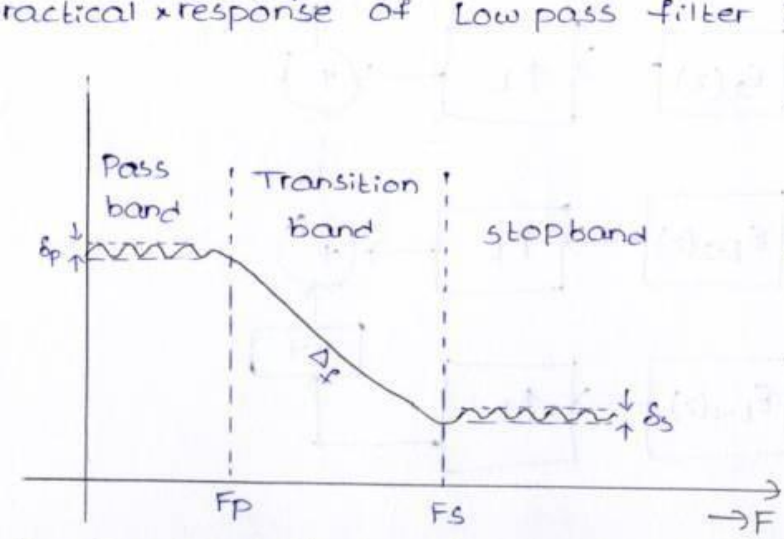
Multi-Stage implementation of interpolator:



where, $I = I_1 \cdot I_2 \cdots\cdots I_l$

⇒ Filter specifications of LPF:

The practical frequency response of Low pass filter is:



Δf = Normalized Transition Bandwidth $= \frac{F_s - F_p}{F_{input\ sampling}}$

where, δ_p - Passband ripple, δ_s - stopband ripple

Fp - Pass band frequency, F_s - stopband frequency

Order (or) length of filter is, $N = \frac{-10\log(\delta_p\delta_s) - 13}{14.6(\Delta f)} + 1$

⇒ Filter specifications for each stage:

→ Pass band Ripple, $\delta_p' = \frac{\delta_p}{I}$

where, I is the number of stages

→ Stop band ~~Ripple~~ $= F_i - F_{stop} \le F \le \frac{F_{i-1}}{2}$

where, $i = 1, 2, 3, \ldots I$

→ Pass band $= 0 \le F \le F_p$

F_{stop} = Edge frequency of transition band

N_i = order or length of the filter $= \frac{-10\log(\delta_p\delta_s) - 13}{14.6(\Delta f_i)} + 1$

→ Implement single stage and two stage decimator for the following specifications.

Sampling rate of i/p signal = 20,000 HZ

Decimator factor, D = 100

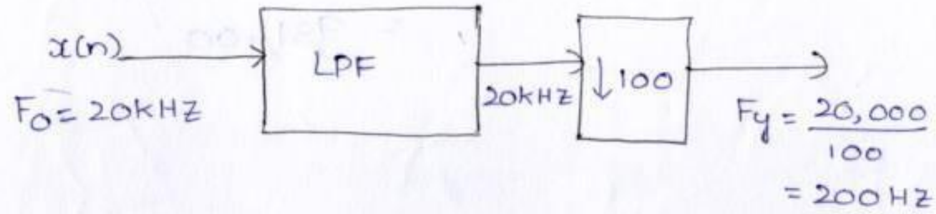
Pass band = 0 to 40HZ

Transition band = 40 to 50HZ

Pass band Ripple = 0.01

Stop band Ripple = 0.002

sol: Single stage implementation:

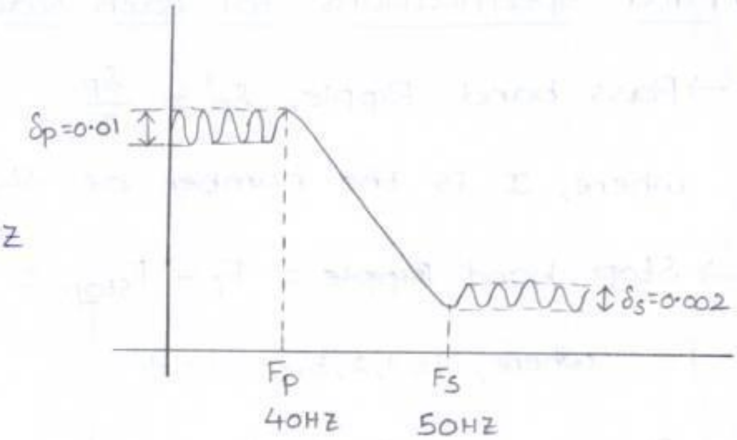


Passband ripple $(\delta_p) = 0.01$

Stopband ripple $(\delta_s) = 0.002$

Passband edge frequency $= F_p = 40 HZ$

Edge frequency of Transition band $= F_s = 50 HZ$



$$\therefore \Delta f = \frac{F_s - F_p}{F_{ins}}$$

$$= \frac{50-40}{F_0}$$

$$= \frac{50-40}{20\times10^3}$$

$$\boxed{\Delta f = 0.0005\ HZ}$$

→ Length (or) order of filter, $N = \dfrac{-10\log(\delta_p\delta_s) - 13}{14.6(\Delta f)} + 1$

$$= \frac{-10\log[(0.01)(0.002)] - 13}{14.6\,(0.0005)} + 1$$

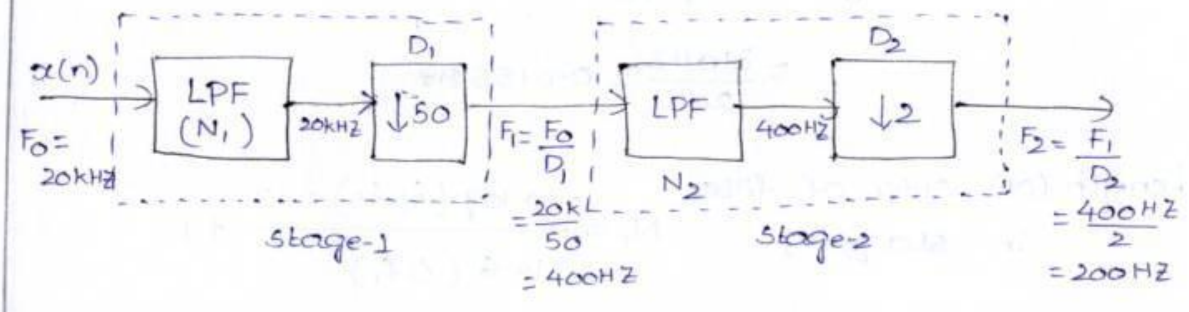
$$\boxed{N = 4657}$$

→ No. of multiplications per second $= \dfrac{N f_0}{D}$

$$= \frac{4657 \times 20k}{100}$$

$$= 931{,}400$$

Two stage implementation:



Design of stage-1 Decimator:

> Note: while implementing in two stage, we can take $D_1 = 25$ & $D_2 = 4$ (or) $D_1 = 20$ & $D_2 = 5$ as our wish. But keep in mind, the highest no. must be in 1st stage & least no. must be in 2nd stage

Passband ripple $= \delta_p' = \frac{\delta p}{I}$

where, I = Number of stages = 2

$$\delta_p' = \frac{\delta p}{2}$$

$$= \frac{0.01}{2} = 0.005$$

Stop band ripple; $\delta_s = 0.002$

Passband edge frequency, $F_p = 40Hz$

WKT, stop band is given as: $F_i - F_{stop} \le F \le \frac{F_{i-1}}{2}$ [∵ i ranges from 1,2,3...I]

$$\Rightarrow F_1 - F_{stop} \le F \le \frac{F_0}{2}$$

But, F_{stop} = edge frequency of given transition band
$= 50 Hz$

$\therefore$ stop band $= 400 - 50 \le F \le \frac{20kHz}{2}$

$= 350\,Hz \le F \le 10kHz$

Though it is given that the edge frequency of transition band is 50Hz, but the range of stopband is from 350Hz to 10kHz.

$\therefore$ Transition band = 40Hz to 350Hz [∵ edge freq of transition band = starting pt of stop band]

$$\Delta f_1 = \frac{F_s - F_p}{F_{i-1}} \cancel{\Delta f_1} = \frac{F_s - F_p}{F_0} = \frac{350-40}{\cancel{F}\ 20k}$$

$$= \frac{310\,HZ}{20k} = 0.0155 \cancel{HZ}$$

→ Length (or) order of filter in stage-1,

$$N_1 = \frac{-10\log(\delta_p \delta_s) - 13}{14.6\,(\Delta f_1)} + 1$$

$$= \frac{-10\log(0.005 \times 0.002) - 13}{14.6(0.0155)} + 1$$

$$\boxed{N_1 = 164.49}$$

→ No. of multiplications per second $= \frac{N_1 f_0}{D_1}$

$$= \frac{164.49 \times 20k}{50}$$

$$= 65,799.9 \ \cancel{\approx 65,78}$$

<u>Design</u> <u>of</u> <u>Stage-2</u> <u>Decimator</u>:

Pass band ripple, $\delta_p' = \frac{\delta_p}{2} = \frac{0.01}{2} = 0.005$

Stop band ripple, $\delta_s = 0.02$

Passband edge frequency $= F_p = 40\,HZ$

Edge frequency of given transition band $= F_s = 50\,HZ$

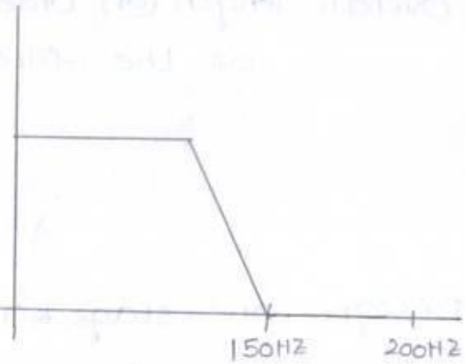
Stop band $= F_i - F_{stop} \le F \le \frac{F_{i-1}}{2}$

$$= F_2 - \cancel{F_s}\, F_{stop} \le F \le \frac{F_1}{2}$$

$$= 200 - 50 \le F \le \frac{400\,HZ}{2}$$

$$= 150\,HZ \le F \le 200\,HZ$$

But transition band is given as 40 to 50HZ. Hence band edge frequency should be taken as 50HZ instead of 150HZ.



$\therefore$ Stop band = $50HZ \le F \le 200HZ$

$\therefore$ Transition band = 40HZ to 50HZ

$$\Delta f_2 = \frac{F_s - F_p}{\cancel{40}\, F_0} = \frac{50-40}{F_1}$$

$$= \frac{50-40}{400}$$

$$= 0.025\ HZ$$

$\rightarrow$ Length (or) Order of filter in stage-2, $N_2 = \dfrac{-10\log(\delta_p \delta_s) - 13}{14.6(\Delta f_2)} + 1$

$$= \frac{-10\log(0.005 \times 0.02) - 13}{14.6(0.025)} + 1$$

$$= 74.97\ HZ$$

$\rightarrow$ No. of multiplications Per second $= \dfrac{N_2 f_1}{D_2}$

$$= \frac{74.97 \times \cancel{20k}\ 400}{2}$$

$$= \cancel{749726}\ 14994$$

$\therefore$ Total No. of multiplications Per sec in cascaded s/m = No. of Multiplications in stage-1 Per Sec + No. of Multiplications Per sec in stage-2

$= 65,799.9 + \cancel{749726}\ 14994$

$= 80793.9$

Overall length (or) order of the filter $= N_1 + (N_2 \times D_1) + D_2$

$= 164.49 + (\overset{74.97}{\cancel{14994}} \times 50) + 2$

$= \cancel{749866.49}\ 3914.99$

→ Design one-stage & Two-stage interpolator to meet the following specifications:

$I = 20$

Pass band: $0 \le F \le 90$

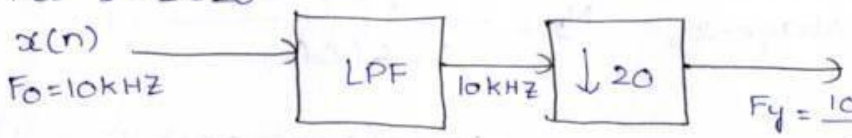
Transition band: $90 \le F \le 100$

Input sampling rate = 10 KHZ

Ripples: $\delta_p = 10^{-2}$, $\delta_s = 10^{-3}$

Sol: Single stage implementations of decimator:

Let $D = I = 20$



$F_y = \frac{10,000}{20}$

$= 500HZ$

Pass band ripple $(\delta p) = 10^{-2} = 0.01$

stopband ripple $(\delta s) = 10^{-3} = 0.001$

Passband edge frequency $= F_p = 90HZ$

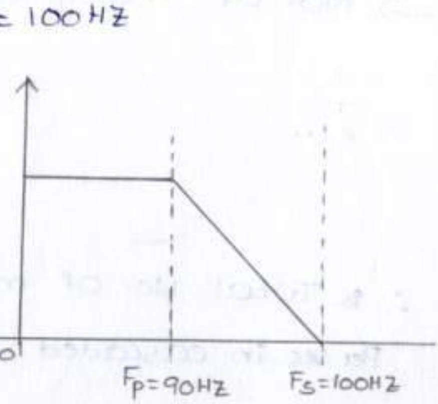
Edge frequency of transition band $= F_s = 100HZ$

$\therefore \Delta f = \frac{F_s - F_p}{F_{ins}}$

$= \frac{100-90}{10k}$

$= \frac{10}{10k}$

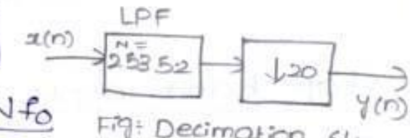
$\boxed{\Delta f = 0.001\ \cancel{Hz}}$



→ Length (or) order of filter, $N = \dfrac{-10\log(\delta_p\delta_s)-13}{14.6(\Delta f)} + 1$

$$= \frac{-10\log(0.01\times 0.001)-13}{14.6(0.001)} + 1$$

$$\boxed{N = 2535.2}$$



→ No. of multiplications per second $= \dfrac{N f_0}{D}$

Fig: Decimation S/m

$$= \frac{2535.2\times 10K}{20}$$

$$= 1,267,623.2$$

Single stage interpolator:

To design interpolator, initially designed decimator and then transposed decimator system produce is interpolator system.

The transposed form of decimation s/m is:

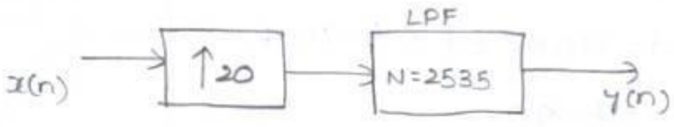
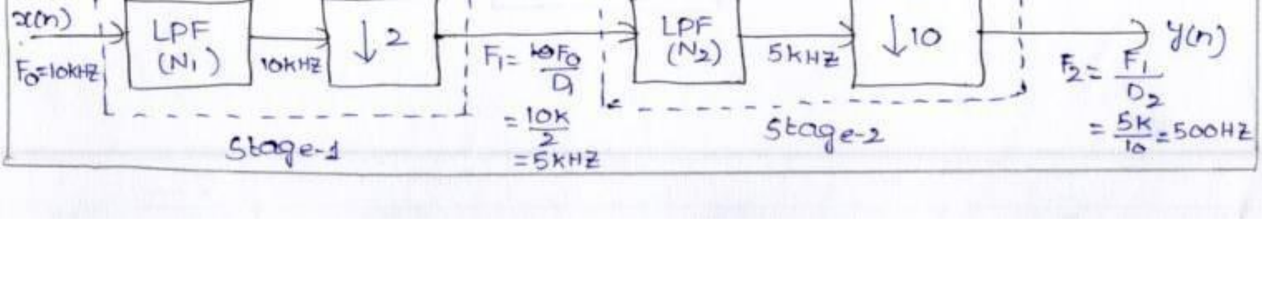


Fig: Interpolator System

Two stage interpolator:

To design two stage interpolator, initially design decimator and then transposed decimator system produces interpolator system.

Let, $D = 20 \Rightarrow D_1 D_2 = 2\times 10 \Rightarrow D_1 = 2,\ D_2 = 10.$



Design of Stage-1 Decimator:

Pass-band ripple $= \delta_p' = \frac{\delta_p}{I}$

where, I = no. of stages = 2

$$\Rightarrow \delta_p' = \frac{\delta_p}{2} = \frac{0.01}{2} = 0.005$$

Stop band ripple, $\delta_s = 0.001$

Passband edge frequency, $F_p = 90\,HZ$

WKT, stopband is given as: $F_i - F_{stop} \leq F \leq \frac{F_{i-1}}{2}$

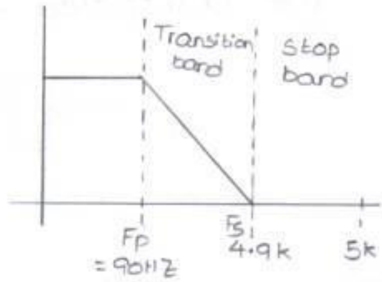
$$\Rightarrow F_1 - F_{stop} \leq F \leq \frac{F_0}{2}$$

But F_{stop} = edge frequency of trasition band

$F_S = 100\,HZ$

$\therefore$ stop band $= 5000 - 100 \leq F \leq \frac{10k}{2}$

$= 4900 \leq F \leq 5000\,HZ$



$\therefore$ Transition band $= 90\,HZ \leq F \leq 4.9\,kHZ$

$$\Delta f_1 = \frac{F_S - F_P}{F_0} = \frac{4.9k - 90}{10k}$$

$$= 0.481$$

→ Length (or) order of filter in stage-1,

$$N_1 = \frac{-10\log(\delta_p'\delta_s) - 13}{14.6(\Delta f_1)} + 1$$

$$= \frac{-10\log(0.005 \times 0.001) - 13}{14.6(0.481)} + 1$$

$$\boxed{N_1 = 6.69}$$

→ No. of multiplications per second $= \frac{N_1 f_0}{D_1}$

$$= \frac{6.69 \times 10k}{2}$$

$$= 33,450$$

Design of stage-2 decimator:

Passband ripple, $\delta p' = \frac{\delta p}{2I} = 0.005$ where, $I=2$ represents two stages

Stop band ripple, $\delta_s = 0.001$

Passband edge frequency, $F_p = 90\,HZ$

$$\text{Stop band} = F_i - F_{stop} \le F \le \frac{F_{i-1}}{2}$$

$$= F_2 - F_{stop} \le F \le \frac{F_{2-1}}{2}$$

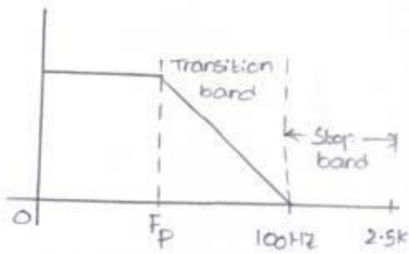
$$= F_2 - F_{stop} \le F \le \frac{F_1}{2}$$

But F_{stop} = Transition band edge frequency

$= 100\,HZ$

$\therefore$ Stop band $= 500\,HZ - 100\,HZ \le F \le \frac{5k}{2}$

$= 400\,HZ \le F \le 2.5\,kHZ$



$\therefore$ Stopband freq $= 100\,HZ \le F \le 2.5\,kHZ$

Transition band $= 90\,HZ \le F \le 100\,HZ$

$$\Delta f_1 = \frac{F_s - F_p}{F_{0_1}} = \frac{100-90}{5k} = 0.002$$

→ Length or order of filter in stage-2, N_2 $= \frac{-10 \log(\delta p' \delta_s) - 13}{14.6(\Delta f_1)}$

$$= \frac{-10 \log(0.005 \times 0.001) - 13}{14.6(0.002)}$$

$$N_2 = 1370.2$$

→ No. of multiplications per second $= \frac{N_2 f_0}{D_2}$

$= \frac{1370.2 \times 10k}{10}$

$= 1,370,215.7$

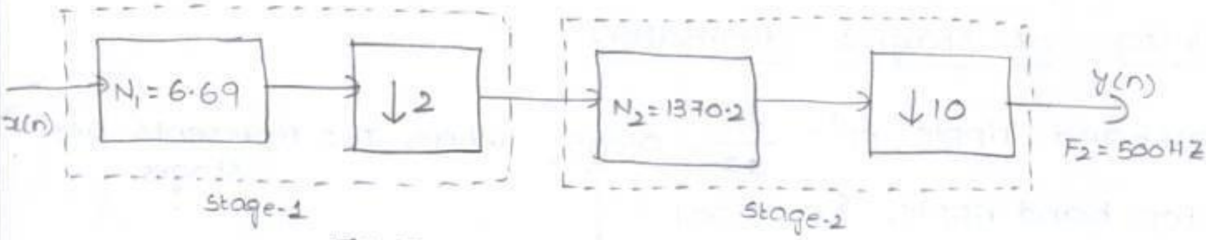


Fig: Two stage decimator system

The transposed form of above decimator system is:

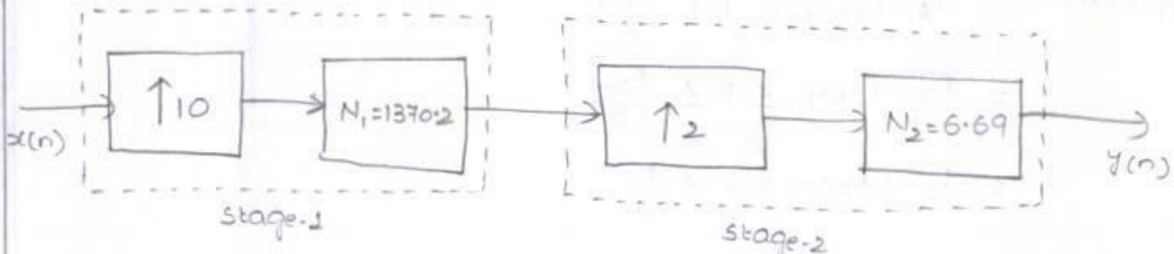


Fig: Two stage Interpolator system

1 Write any two applications of multirate DSP ?

Implementation of a narrow band LPF:

A narrow band low pass filter, is characterized by a narrow pass band and narrow transition band. It requires a very large number of coefficients due to high value of N, it is suspective to finite word length effects. In addition the number of computations and memory location required are very high.

To overcome these problems, multirate approach is used in implementing a narrow band low pass filter. The cascading stage of a decimator and interpolator are LPF. The input signal is first passed through a LPF. The sampling frequency F_{st} the input sequence $x(n)$ is first reduced by the some factor D and then again LPF is performed.

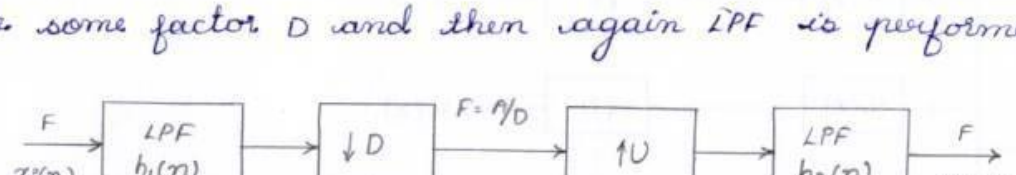


Fig: Narrow BPF

To meet the desired specifications of narrow band LPF, the filters $h_1(n)$ and $h_2(n)$ should be identical with same pass band ripple $\delta_p/2$ and the same stop band ripple δ_s

Filter banks [and analysis]:

The D-channel analysis filter bank is shown in figure. It consists of 'D' sub filters. All the sub-filters are equally spaced in frequency and each have the same bandwidth, the spectrum of the input signal lies in the range $0 \le \omega \le \pi$.

The filter bank splits the signal into a number of sub-bands each having a bandwidth π/D. The filter $H_0(z)$ is low pass. $H_1(z)$ to $H_{D-2}(z)$ are band pass and $H_{D-1}(z)$ is high pass. As the spectrum of the signal is band limited to π/D, the sampling rate can be reduced by a factor D. The down sampling moves all the pass band signals to a base band $0 \le \omega \le \pi/D$.

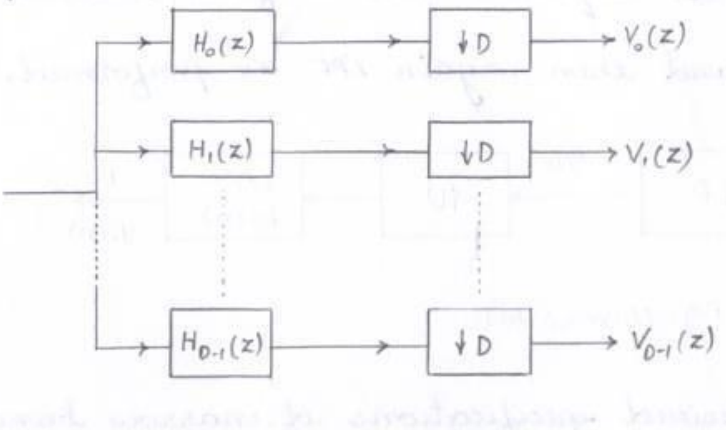


Fig: Analysis filter bank.

2 compare the single-stage, two stage decimator with the following specifications.

Sampling rate of a signal has to be reduced from 10 kHz to 500 Hz. The filter of decimator has pass band edge 150 Hz, stop band edge of 180 Hz, pass band ripple = 0.002 and stop band ripple = 0.001.

Solution :

Single stage implementation :

$\delta_s = 0.001$

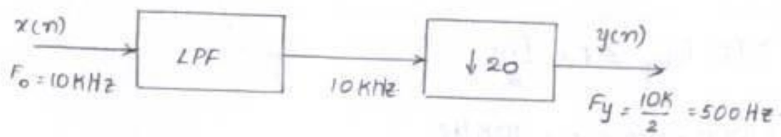
$\delta_p = 0.002$

$F_p = 150\,Hz$

$F_s = 180\,Hz$.

$D = \frac{10K}{500}$

$= 20$



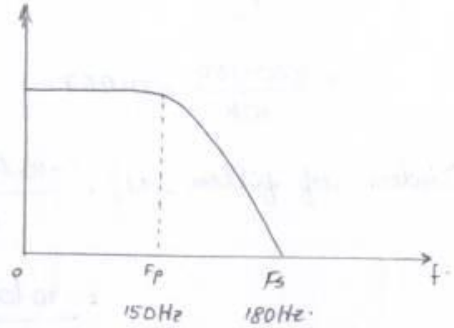
Hence the decimation factor is 20.

The length (N) or order of filter $= \frac{-10\log(\delta_p\delta_s)-13}{14.6\cdot\Delta F} + 1$

$\therefore \Delta F = \frac{F_s - F_p}{F_{in}}$

$= \frac{180-50}{10KHz}$

$= 0.003\,Hz$



$\Rightarrow N = \frac{-10\log\delta_p\delta_s - 13}{14.6\cdot\Delta F} + 1$

$= \frac{-10\log(0.002\times 0.001)-13}{14.6\times 0.003} + 1$

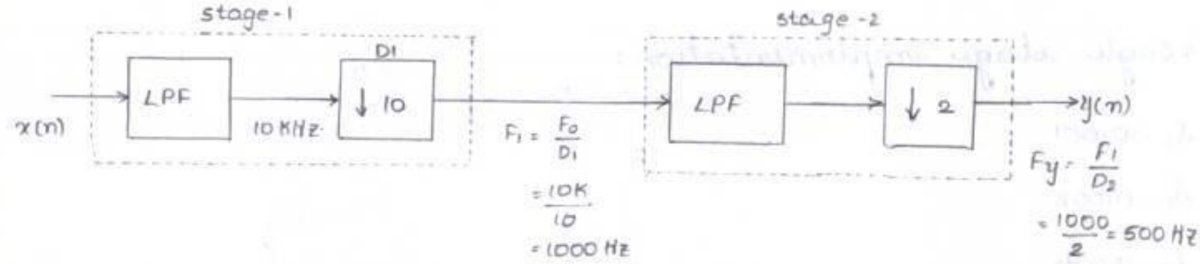
$= 1005.33$.

Number of multiplications per second $= \frac{NF_o}{D}$

$= \frac{1005.33 \times 10K}{20}$

$= 502.665$

Two stage implementation:



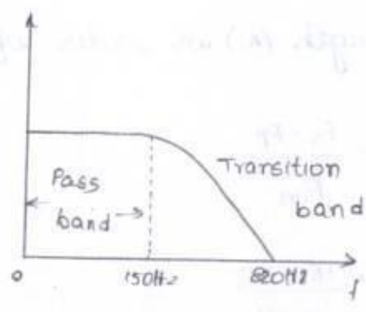
Pass band ripple $= \delta_p' = \frac{\delta p}{I} = \frac{0.002}{2} = 0.001$

stop band $= F_i - F_{stop} \le F \le \frac{F_{i-1}}{2}$

$= 1000 - 180 \le F \le \frac{10KHz}{2}$

$= 820 \le F \le 5KHz$

Transition band $= 150\,Hz \le F \le 820\,Hz$



$\therefore \Delta F_1 = \frac{820 - 150}{F_o}$

$= \frac{820-150}{10K} = 0.067$

Order of filter, $N_1 = \frac{-10 \log (\delta_p \delta_s) - 13}{14.6 (\Delta F_1)} + 1$

$= \frac{-10 \log (0.001 \times 0.001)}{14.6 (0.067)} + 1$

$\therefore N_1 = 49$

Number of multiplications per second $= \frac{N_1 F_o}{D} = \frac{49 \times 10K}{10}$

$= 49000$

Design of stage 2 :

Pass band ripple , $\delta_p' = \frac{\delta_P}{2} = \frac{0.002}{2} = 0.001$

$\delta_S = 0.001$

$F_p = 150\,Hz$

$F_S = 180\,Hz$

Stop band $= F_i - F_{stop} \leq F \leq \frac{F_{i-1}}{2}$

$= F_2 - F_{stop} \leq F \leq \frac{1000}{2} = 500 - 180 \leq F \leq \frac{1000}{2}$

$\therefore$ Stop band $= 320 \leq F \leq 500\,Hz$

$\therefore$ Transition band is given as 320Hz. Hence stop band edge frequency should be taken as 180Hz instead of 320Hz.

Hence, stop band $= 180\,Hz \leq F \leq 500\,Hz$

Transition band = 150 to 180 Hz

$$\Delta F_2 = \frac{180-150}{F_1} = \frac{30}{1000} = 0.03$$

Length of filter , $N_2 = \frac{-10 \log(\delta_p' \delta_s) - 13}{14.6(\Delta F_2)} + 1$

$$= \frac{-10 \log(0.001 \times 0.001) - 13}{14.6(0.03)} + 1$$

$$N_2 = 108$$

Number of multiplications per second $= \frac{N_2 F_1}{D_2}$

$$= \frac{108 \times 1000}{2} = 54000$$

Total no. of multiplications per second in cascaded system = no. of multiplications in stage 1 per second + No. of multiplications in stage 2 per second.

$= 49000 + 54000$

$= 103000$

$\therefore$ Total length (or) order of filter $= N_1 + (N_2 \times D_1) + D_2$

$= 49 + (108 \times 10) + 2$

$= 1131$

3 Design a linear phase FIR filter that satisfies following specifications based on single and two stage multi-rate structure.

Sampling rate $(F_0) = 10KHz$, pass band $= 0 \leq F \leq 60$, transition band $= 60 \leq F \leq 65$, pass band ripple $(\delta_p) = 10^{-1}$, stop band $(\delta_s) = 10^{-3}$

given data is

$F_0 = 10KHz$

$F_{stop} \leq \frac{F_{in}}{2D}$

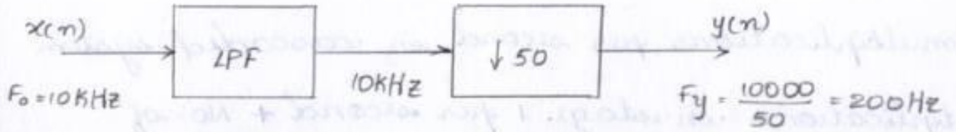
$65 \leq \frac{10K}{2D}$

$\Rightarrow 65 \leq \frac{5K}{D}$

$\Rightarrow D \leq 76.9$

Lets assume $D = 50$

Single stage implementation :



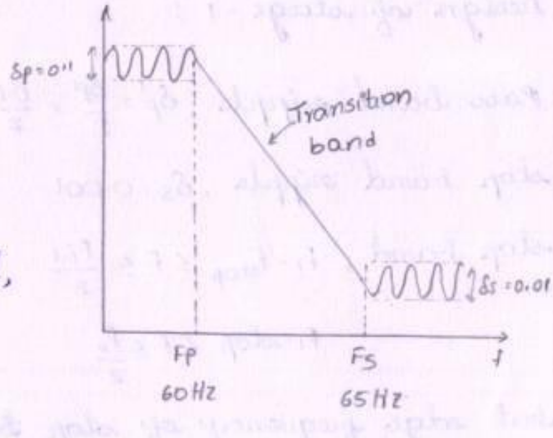
Pass band ripple $\delta_p = 0.1$

stop band ripple $\delta_s = 0.001$

Pass band edge frequency $= 60Hz$

Edge frequency of transition band,

$F_s = 65\,Hz$



$$\therefore \Delta F = \frac{F_s - F_p}{F_{in}}$$

$$= \frac{65-60}{10K} = 0.0005\,Hz$$

length (or) order of the filter, $N = \dfrac{-10\log(\delta p \cdot \delta s) - 13}{14.6(\Delta f)}$

$$= \frac{-10\log(0.1 \times 0.01) - 13}{14.6 \times 0.005} + 1$$

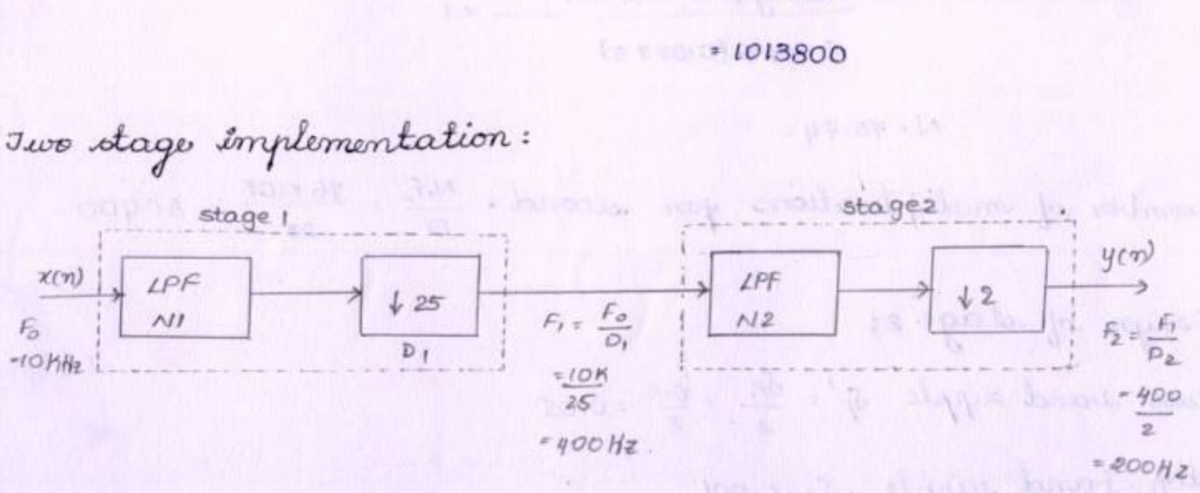
$$N = 5069.5$$

No. of multiplications per second $= \dfrac{NF_s}{D}$

$$= \frac{5069.5 \times 10K}{50}$$

$$= 1013800$$

Two stage implementation:



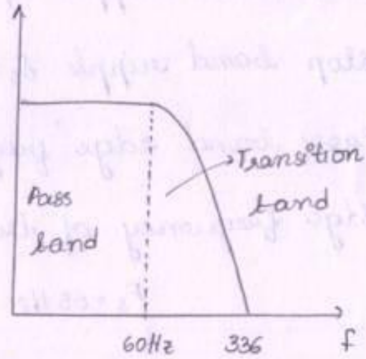
Design of stage - 1 :

Pass band ripple , $\delta_p' = \frac{\delta_p}{I} = \frac{0.1}{2} = 0.05$

stop band ripple , $\delta_s = 0.001$

stop band ; $F_i - F_{stop} \leq F \leq \frac{F_{i-1}}{2}$

$$F_i - stop \leq F \leq \frac{F_o}{2}$$



but edge frequency of stop transition band is

$$F_{stop} = 65 \text{ Hz}$$

$$\therefore \text{stop band} = 400 - 65 \leq F \leq \frac{10K}{2}$$

$$= 335 \leq F \leq 5 \text{ KHz}$$

Transition band = 60Hz to 335 Hz

$$\therefore \Delta F_1 = \frac{335 - 60 \text{ Hz}}{F_o} = \frac{275}{10K}$$

$$= 0.0275$$

Order of filter $N_1 = \frac{-10 \log(\delta_p \delta_s) - 13}{14.6 (\Delta F_1)} + 1$

$$= \frac{-10 \log(0.05 \times 0.001) - 13}{14.6 (0.0275)} + 1$$

$$N = 75.74.$$

Number of multiplications per second = $\frac{N_1 F_o}{D} = \frac{76 \times 10K}{25} = 30400$

Design of stage 2:

Pass band ripple $\delta_p' = \frac{\delta_{P_1}}{2} = \frac{0.1}{2} = 0.05$

stop - band ripple , $\delta_s = 0.001$

Pass band edge frequency $F_p = 60 \text{ Hz}$

$F_{stop} = 65\ Hz$

$\therefore$ Stop band $= F_i - F_{stop} \le F \le \frac{F_{i-1}}{2}$

$= F_2 - F_{stop} \le F \le \frac{F_{2-1}}{2}$

$= 200 - 65 \le F \le \frac{400}{2}$

$= 135 \le F \le 200$

But transition band $= 60 - 65 = 5$

Hence stop band edge frequency is 65 Hz instead of 135 Hz

$\therefore$ stop band $= 65\ Hz \le F \le 200\ Hz$

Transition band $= 60\ Hz$ to $65\ Hz$

$\therefore \Delta F_2 = \frac{65-60}{F_1} = \frac{5}{400} = 0.0125$

Order of filter $N_2 = \frac{-10 \log(\delta p \cdot \delta s) - 13}{14.6(\Delta F_2)} + 1$

$= \frac{-10 \log(0.05 \times 0.001)}{14.6(0.0125)} + 1$

$N_2 = 165$

Number of multiplications per second $= \frac{N_2 F_1}{D_2}$

$= \frac{165 \times 400}{2}$

$= 33000$

Total number of multiplications per second in cascaded system = no. of multiplications per second in stage 1

+ no. of multiplications per second in stage 2

$= 30400 + 33000$

$= 63400$

Over all length (or) order of filter

$= N_1 + (N_2 \times D_1) + D_2$

$= 76 + (165 \times 25) + 2$

$= 4203$