STATE FEEDBACK CONTROLLERS AND OBSERVERS

POLE PLACEMENT DESIGN

The concept of controllability is the basic for the solutions of the pole placement problem and the concept of observability plays an important role for the design of state observers. The design method based on the pole placement coupled with state observers is one of the fundamental design methods available to control engineers. If the system is completely state controllable, then the desired closed loop poles in the z-plane can be selected and the system that will give such closed loop poles can be designed. The design approach of placing the closed loop poles in the desired locations in the z-plane is called the pole placement design technique; that is, in the pole placement design technique we fed back all state variables so that all poles of the closed loop system are placed at the desired locations. In practical control systems, however measurement of all state variables may not possible; in that case, not all the state variables will be available for feedback. To implement a design based on state feedback; it becomes necessary to estimate the immeasurable state variables. Such estimation can be done by use of state observers.

NECESSARY AND SUFFICIENT CONDITION FOR ARBITRARY POLE PLACEMENT

The state equation of the open loop system is

$$x(k+1) = Gx(k) + Hu(k)$$

Where $x(k)$ = State vector at the kth sampling instant

$u(k)$ = Control signal at the kth sampling instant

G = n×n matrix

H = n×1 matrix

If we are assuming the unbounded control signal as $u(k) = -Kx(k)$

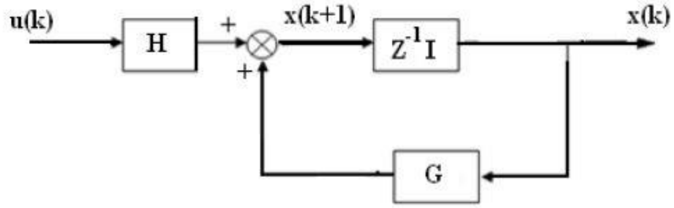
Where $K = [k_1 k_2 k_3 \ldots k_n]$ = State feedback gain matrix

Then, the system becomes closed loop control system, and its state equation becomes

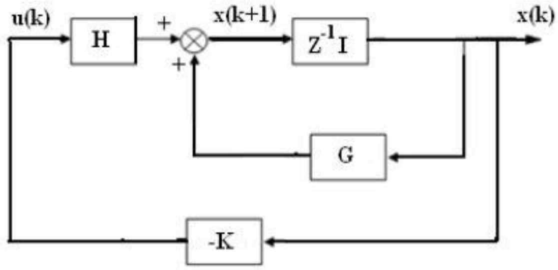
$$x(k+1) = Gx(k) + H \times -Kx(k)$$

$$= (G - HK)x(k)$$

Note that we choose matrix K such that the eigen values of $G - HK$ are the desired closed loop poles; $\mu_1, \mu_2, \mu_3 ....\mu_n$.



(a): Open-loop control system



(b): Closed-loop control system

Fig : Block diagram of feed-back gain matrix

A necessary and sufficient condition for arbitrary placement of closed loop eigen values is that the system given by eqn is completely state controllable.

Necessary Condition:

The necessary condition for the system is, if the system is not completely state controllable, and then there are eigen values of $G - HK$ that cannot be controlled by state feedback. We will prove this condition now.

Suppose the system of equations is not completely state controllable, then the rank of the controllability matrix is less than 'n' (or)

$$Rank \ [H : GH : ...G^{n-1}H] = q < n$$

This means that there are 'q' linearly independent column vectors in the controllability matrix. Lets us define such 'q' linearly independent column vectors as $f_1, f_2, f_3 ....f_q$ .Also, let us choose n-q additional n-vectors $v_{q+1}, v_{q+2}, v_{q+3} ....v_n$ such that

$P = [f_1 : f_2 : .... : f_q : v_{q+1} : v_{q+2} : .... : v_n]$ is of rank n.

By using matrix P as the transformation matrix, let us define

$P^{-1}GP = \hat{G}$ and $P^{-1}H = \hat{H}$

$$\Rightarrow GP = P\hat{G}$$

$$[Gf_1 : Gf_2 : \ldots\ldots : Gf_q : Gv_{q+1} : Gv_{q+2} : \ldots\ldots\ldots Gv_n] = [f_1 : f_2 : \ldots\ldots f_q : v_{q+1} : v_{q+2} : \ldots\ldots v_n]\hat{G}$$

$$H = P\hat{H} = [f_1 : f_2 : \ldots\ldots\ldots : f_q : v_{q+1} : v_{q+2} : \ldots\ldots\ldots : v_n]\hat{H}$$

Since we have here 'q' linearly independent column vectors $f_1, f_2, f_3 \ldots\ldots f_q$ we can use Caley-Hamilton theorem to express matrices $Gf_1, Gf_2, Gf_3 \ldots\ldots Gf_q$ in terms of these 'q' vectors. i.e.

$$\begin{aligned}
Gf_1 &= g_{11} f_1 + g_{2g} f_2 + \ldots + g_{q1} f_q \\
Gf_2 &= g_{12} f_1 + g_{22} f_2 + \ldots + g_{q2} f_q \\
\vdots & \qquad\qquad \vdots \\
Gf_q &= g_{1q} f_1 + g_{2q} f_2 + \ldots + g_{qq} f_q
\end{aligned}$$

Hence, equation can be written as

$$[Gf_1 : Gf_2 : \ldots : Gf_q : Gv_{q+1} : Gv_{q+2} : \ldots : Gv_n]$$

$$= [f_1 \cdots\cdots f_q : v_{q+1} \cdots\cdots v_n] \left[\begin{array}{cccc|cccc}
g_{11} & g_{12} & \cdots & g_{1q} & g_{1q+1} & g_{1q+2} & \cdots & g_{1n} \\
g_{21} & g_{22} & \cdots & g_{2q} & g_{2q+1} & g_{2q+2} & \cdots & g_{2n} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
g_{q1} & g_{q1} & \cdots & g_{qn} & g_{qq+1} & g_{qq+2} & \cdots & g_{qn} \\
\hline
0 & 0 & \cdots & 0 & g_{q+1q+1} & g_{q+1q+2} & & g_{q+1n} \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
\vdots & & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & g_{nq+1} & g_{nq+2} & \cdots & g_{nn}
\end{array}\right].$$

To simplify the notations, let us define

$$\begin{bmatrix}
g_{11} & g_{12} & \cdots & g_{1q} \\
g_{21} & g_{22} & \cdots & g_{2q} \\
\vdots & \vdots & & \\
g_{q1} & g_{q2} & \cdots & g_{qq}
\end{bmatrix} = G_{11}$$

$$\begin{bmatrix}
g_{1q+1} & g_{1q+2} & \cdots & g_{1n} \\
g_{2q+1} & g_{2q+2} & \cdots & g_{2n} \\
\vdots & \vdots & & \\
g_{qq+1} & g_{qq+2} & \cdots & g_{qn}
\end{bmatrix} = G_{12}$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = G_{21} = (n-q)\times q = \text{Zero matrix}$$

$$\begin{bmatrix} g_{q+1q+1} & g_{q+1q+2} & \cdots & g_{q+1n} \\ g_{q+2q+1} & g_{q+2q+2} & \cdots & g_{q+2n} \\ \vdots & \vdots & & \\ g_{nq+1} & g_{nq+2} & \cdots & g_{nn} \end{bmatrix} = G_{22}$$

Then eq. (8.6) can be written as follows

$$[Gf_1 \ldots \cdots Gf_q : Gv_{q+1} \cdots\cdots Gv_n] = [f_1 \cdots\cdots f_q : v_{q+1} \cdots\cdots v_n] \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix}$$

Thus, $\hat{G} = \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix}$

∴ we have

$$H = [f_1 : f_2 \cdots\cdots f_q : v_{q+1} : v_{q+2} \cdots\cdots v_n]\, \hat{H}$$

H can be written in terms of 'q' linearly independent column vectors $f_1, f_2, \ldots f_q$. Thus we have $H = [h_{11}f_1 + h_{21}f_2 + \cdots\cdots + h_{q1}f_q]$

$$h_{11}f_1 + h_{21}f_2 + \ldots\ldots + h_{q1}f = [f_1 \cdots\cdots f_q : v_{q+1} \ldots\ldots v_n] \begin{bmatrix} h_{11} \\ h_{21} \\ \vdots \\ h_{q1} \\ --- \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Thus, $\hat{H} = \begin{bmatrix} H_{11} \\ --- \\ 0 \end{bmatrix}$

Where, $H_{11} = \begin{bmatrix} h_{11} \\ h_{21} \\ \vdots \\ h_{q1} \end{bmatrix}$

The characteristic equation of the closed loop system given by

$$|ZI - G + HK| = 0$$

Let us define

$$\hat{K} = KP = [K_{11} \quad : \quad K_{12}]$$

Where K_{11} is a $1\times$q matrix and $K_{1\ 2}$ is a $1\times$(n-q) matrix.

$$\hat{K} = KP$$

$$\hat{K}P^{-1} = KPP^{-1} = KI = K$$

$$\Rightarrow K = \hat{K}P^{-1} = [K_{11} \quad : \quad K_{12}]P^{-1}$$

Then the characteristic equation for the closed loop system can be written as follows

$$|ZI - G + HK| = |P^{-1}|\,|ZI - G + HK|\,|P|$$

$$= |ZI - P^{-1}GP + P^{-1}HKP|$$

$$= |ZI - \hat{G} + \hat{H}\hat{K}|$$

By substituting

$$|ZI - \hat{G} + \hat{H}\hat{K}| = \left| Z\begin{bmatrix} I_q & 0 \\ 0 & I_{n-q} \end{bmatrix} - \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix} + \begin{bmatrix} H_{11} \\ 0 \end{bmatrix}[K_{11} \quad K_{12}] \right|$$

$$= \begin{vmatrix} ZI_q - G_{11} + H_{11}K_{11} & -G_{12} + H_{11}K_{12} \\ 0 & ZI_{n-q} - G_{22} \end{vmatrix}$$

$$\Rightarrow \left| ZI - \hat{G} + \hat{H}\hat{K} \right| = \left| ZI_q - G_{11} + H_{11}K_{11} \right| \left| ZI_{n-q} - G_{22} \right|$$

Equation shows that matrix $K = \hat{K} P^{-1}$ has control over the 'q' eigen values of $G_{11} + H_{11}K_{11}$, but not over the $n - q$eigen values of $G_{2\,2}$ That is, there are (n-q) eigen values of [G-HK] that do not depend on matrix K. Hence, we have proved that complete state controllability is a necessary condition for controlling all eigen values of matrix [G-HK].

Sufficient Condition:

The sufficient for the system is, if the system is completely state controllable then there exists a matrix K that will place the eigen values of [G-HK] or closed loop poles at the desired locations.

The desired eigen values of [G-HK] are $\mu_1, \mu_2, \mu_3 .... \mu_n$ any complex eigen values are to occur as congugate pairs. Noting that the characteristic equation of the original system is

$$\left| ZI - G \right| = Z^n + a_1 Z^{n-1} + ... + a_{n-1} Z + a_n = 0$$

We define the transformation matrix (T) as follows

$$T = MW$$

Where $M = \left[H \ \vdots \ GH \ \vdots \ \cdots \ \vdots \ G^{n-1}H \right]$

Which is of rank n, and where

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

We can get

$$\hat{G} = T^{-1}GT = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

and

$$\hat{H} = T^{-1}H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Next we define

$$\hat{K} = KT = \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \cdots & \delta_1 \end{bmatrix}$$

$$\hat{H}\hat{K} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \cdots & \delta_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \\ \delta_n & \delta_{n-1} & \cdots & \cdots & \delta_1 \end{bmatrix}$$

The characteristic equation of the system is

$$|ZI - G + HK| = \left|ZI - \hat{G} + \hat{H}\hat{K}\right|$$

$$= \left| Z\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \\ \delta_n & \delta_{n-1} & \cdots & \cdots & \delta_1 \end{bmatrix} \right|$$

$$
= \begin{vmatrix} Z & -1 & \cdots & \cdots & 0 \\ 0 & Z & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & -1 \\ a_n + \delta_n & a_{n-1} + \delta_{n-1} & \cdots & \cdots & Z + a_1 + \delta_1 \end{vmatrix}
$$

$$
= Z^n + (a_1 + \delta_1) Z^{n-1} + \dots + (a_{n-1} + \delta_{n-1}) Z + a_n + \delta_n = 0
$$

The characteristic equation with desired eigen values is given by

$$
(Z - \mu_1)(Z - \mu_2) \dots\dots (Z - \mu_n) = Z^n + \alpha_1 Z^{n-1} + \dots + \alpha_{n-1} Z + \alpha_n = 0
$$

On comparing equations

$$
\left.\begin{aligned} \alpha_1 &= a_1 + \delta_1 \\ \alpha_2 &= a_2 + \delta_2 \\ &\vdots \\ \alpha_n &= a_n + \delta_n \end{aligned}\right\}
$$

From equations

$$
K = \hat{K} T^{-1}
$$

$$
= \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \cdots & \delta_1 \end{bmatrix} T^{-1}
$$

$$
= \begin{bmatrix} \alpha_n - a_n \vdots & \alpha_{n-1} - a_{n-1} \vdots & \cdots & \cdots & \alpha_1 - a_1 \end{bmatrix} T^{-1}
$$

Where a_i 's and α_i 's known co-efficients and T is the known matrix. Hence we have determined the required feedback gain matrix K in terms of known co-efficients and a known matrix of the system. This proves the sufficient condition.

ACKERMAN'S FORMULA

Consider the discreet time system defined by

$$
x(k+1) = Gx(k) + Hu(k)
$$

It is assumed that the system is completely controllable.

By using the feedback, $u(k) = -Kx(k)$, we wish to place closed loop poles at

$Z = \mu_1, Z = \mu_2, \cdots\cdots, Z = \mu_n$, i.e. the desired characteristic equation is

$$|ZI - G + HK| = (Z - \mu_1)(Z - \mu_2)\cdots\cdots(Z - \mu_n)$$

$$= Z^n + \alpha_1 Z^{n-1} + \alpha_2 Z^{n-2} + \cdots\cdots + \alpha_{n-1} Z + \alpha_n = 0$$

Let us define

$$\hat{G} = G - HK$$

Since the Caley-Hamilton Theorem states that $\hat{G}$ satisfies its own characteristic equation.

$$\hat{G}^n + \alpha_1 \hat{G}^{n-1} + \alpha_2 \hat{G}^{n-2} + \cdots\cdots + \alpha_{n-1}\hat{G} + \alpha_n I = \Phi(\hat{G}) = 0$$

We shall utilize this last equation to derive Ackermann's formula.

Consider the following identities

$$I = I$$

$$\hat{G} = G - HK$$

$$\hat{G}^2 = (G - HK)^2 = (G - HK)(G - HK)$$

$$= G^2 - GHK - HKG + H^2K^2 \quad = G^2 - GHK - HK(G - HK)$$

i.e $\hat{G}^2 = G^2 - GHK - HK\hat{G}$

$$\hat{G}^3 = (G - HK)^3 = G^3 - G^2HK - GHK\hat{G} - HK\hat{G}^2$$

$$\vdots$$

$$\vdots$$

$$\hat{G}^n = (G - HK)^n = G^n - G^{n-1}HK - GHK\hat{G} - HK\hat{G}^{n-1}$$

Multiplying the proceeding equation in order by $\alpha_n, \alpha_{n-1}, \ldots\ldots, \alpha_0$ (where $\alpha_0 = 1$) respectively and adding the result we obtain

$$\alpha_n I + \alpha_{n-1} \hat{G} + \alpha_{n-2} \hat{G}^2 + \ldots\ldots + \hat{G}^n = \alpha_n I + \alpha_{n-1} G + \alpha_{n-2} G^2$$

$$+ \ldots + G^n - \alpha_{n-1} HK - \alpha_{n-2} GHK - \alpha_{n-2} HK \hat{G} \ldots - G^{n-1} HK \ldots - HK \hat{G}^{n-1}$$

$$\Rightarrow \phi(\hat{G}) = \phi(G) - \alpha_{n-1} HK - \alpha_{n-2} GHK - \alpha_{n-2} HK \hat{G} \cdots\cdots - HK \hat{G}^{n-1} - G^{n-1} HK$$

$$= \phi(G) - \left[H \;\vdots\; GH \;\vdots\; \cdots \;\cdots\; \vdots G^{n-1} H\right] \begin{bmatrix} \alpha_{n-1} K + \alpha_{n-2} K \hat{G} + \ldots\ldots + K \hat{G}^{n-1} \\ \alpha_{n-2} K + \alpha_{n-3} K \hat{G} + \ldots\ldots + K \hat{G}^{n-2} \\ \vdots \\ \vdots \\ K \end{bmatrix}$$

Note that $\phi(\hat{G}) = 0$

The above equation can be written as

$$\phi(G) = \left[H \;\vdots\; GH \;\vdots\; \cdots \;\cdots\; \vdots G^{n-1} H\right] \begin{bmatrix} \alpha_{n-1} K + \alpha_{n-2} K \hat{G} + \ldots\ldots + K \hat{G}^{n-1} \\ \alpha_{n-2} K + \alpha_{n-3} K \hat{G} + \ldots\ldots + K \hat{G}^{n-2} \\ \vdots \\ \vdots \\ K \end{bmatrix}$$

Since the system is completely state controllable, the controllability matrix

$\left|H \;\vdots\; GH \;\vdots\; \cdots \;\cdots\; \vdots G^{n-1} H\right|$

is of rank ‘n’ and its inverse exists.

$$\begin{bmatrix} \alpha_{n-1} K + \alpha_{n-2} K \hat{G} + \ldots\ldots + K \hat{G}^{n-1} \\ \alpha_{n-2} K + \alpha_{n-3} K \hat{G} + \ldots\ldots + K \hat{G}^{n-2} \\ \vdots \\ \vdots \\ K \end{bmatrix} = \left|H \;\vdots\; GH \;\vdots\; \cdots \;\cdots\; G^{n-1} H\right|^{-1} \phi(G)$$

Pre-multiplying both sides of this last equation by $\begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix}$

We obtain $\begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix}\begin{bmatrix}\alpha_{n-1}K+\alpha_{n-2}K\hat{G}+\ldots\ldots+K\hat{G}^{n-1}\\ \alpha_{n-2}K+\alpha_{n-3}K\hat{G}+\ldots\ldots+K\hat{G}^{n-2}\\ \vdots \\ \vdots \\ K\end{bmatrix}$

$$=\begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix}\begin{bmatrix}H & \vdots & GH & \vdots & \cdots & \cdots & G^{n-1}H\end{bmatrix}^{-1}\phi(G)$$

Equation gives the required state feedback matrix K.

Problem-1: Consider the system $x(k+1)=Gx(k)+Hu(k)$ where $G=\begin{bmatrix}0 & 1\\ -016 & -1\end{bmatrix}$ $H=\begin{bmatrix}0\\ 1\end{bmatrix}$

Determine a suitable state feedback gain matrix K such that the system will have closed loop poles at $Z=0.5+j0.5 \quad Z=0.5-j0.5$

Solution: Given that

$$G=\begin{bmatrix}0 & 1\\ -016 & -1\end{bmatrix} H=\begin{bmatrix}0\\ 1\end{bmatrix}$$

Method-1

$$GH=\begin{bmatrix}0 & 1\\ -016 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$Q_c=\begin{bmatrix}H & \vdots & GH\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}$$

$|Q_c|\neq 0$, hence rank=2=n. Thus the system is completely state controllable, therefore arbitrary pole placement is possible.

$$|ZI-G|=\begin{vmatrix}Z & -1\\ 0.16 & Z+1\end{vmatrix}=Z^2+Z+0.16=0$$

$$\Rightarrow a_1=1, a_2=0.16$$

The characteristic equation for the desired system is

$$|ZI - G + HK| = (Z - 0.5 - j0.5)(Z - 0.5 + j0.5)$$

$$= (Z - 0.5)^2 + 0.5^2$$

$$= Z^2 - Z + 0.5 = 0$$

$$\Rightarrow \alpha_1 = -1, \alpha_2 = 0.5$$

$$T = MW = [H \; \vdots \; GH]\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Notice that the original system is already in a controllable canonical form and therefore the transformation matrix T becomes I

$$\therefore K = [\alpha_2 - a_2 \; \vdots \; \alpha_1 - a_1]I$$

$$= [0.5 - 0.16 \; \vdots \; -1 - 1]$$

$$= [0.34 \; \vdots \; -2]$$

According to Ackermann's formula

$$K = [0 \quad 1] \; [H \quad GH]^{-1} \phi(G)$$

$$[H \quad GH]^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\phi(G) = G^2 - G + 0.5I$$

$$= \begin{bmatrix} -0.16 & -1 \\ 0.16 & 0.84 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.16 & 1 \end{bmatrix} + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.34 & -2 \\ 0.32 & 2.34 \end{bmatrix}$$

Thus, $K = [0 \quad 1] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.34 & -2 \\ 0.32 & 2.34 \end{bmatrix} = [1 \quad 0] \begin{bmatrix} 0.34 & -2 \\ 0.32 & 2.34 \end{bmatrix} = [0.34 \quad -2]$

FULL-ORDER OBSERVER

A state observer or a state estimator is a subsystem in the control system that performs an estimation of state variables based on the measurements of the output and control variables.

Fig: Schematic diagram of the state observer

The state observer will have y (k) and u (k) as inputs and $\hat{x}(k)$ as output as shown in fig

A full order state observer means that we observe or estimate all n state variables regardless of whether some state variables are available for direct measurement. The order of the state observer will be same as that of the system. Observation of only the immeasurable state variables is referred to as minimum order state observer or reduced order state observer.

DESIGN OF FULL-ORDER OBSERVER

Consider

$$x(k+1) = Gx(k) + Hu(k)$$

$$y(k) = Cx(k)$$

Where x(k) = State vector

y(k) = Output vector

u(k) = Control Vector

G = n×n non-singular matrix

H = n×r matrix

C = 1×n matrix

The system is assumed to be completely state controllable and completely observable. Thus the inverse of $\left[G^T \vdots \;\; G^T C^T \vdots \;\; \cdots \;\; \cdots \;\; \left(G^T\right)^{n-1} C^T\right]$ exists.

The control law to be assumed as

$u(k) = -K\hat{x}(k)$

Where $\hat{x}(k)$ is the observed state.

The system configuration is as shown in the figure

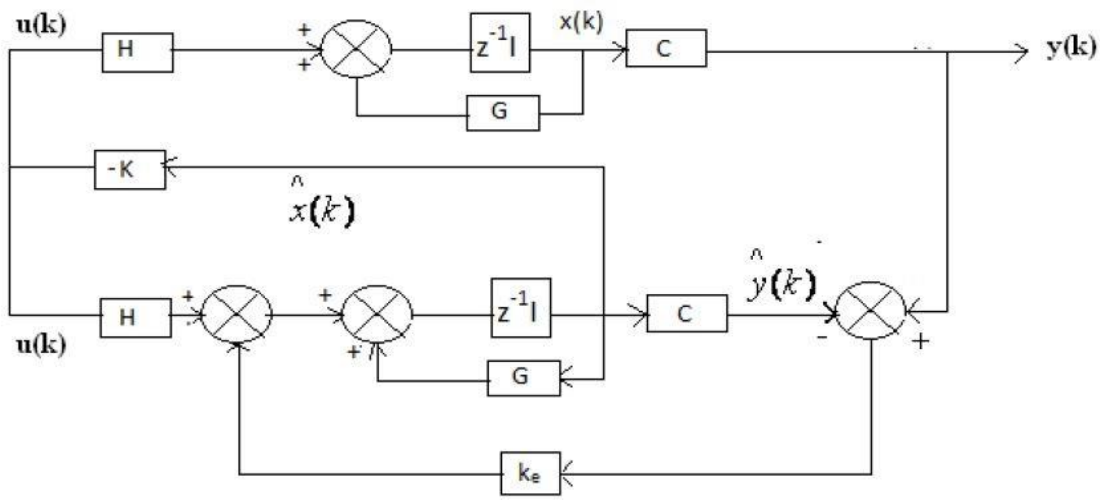


Fig: Observed state feedback control system

The state observer dynamics are given by the equation

$$\hat{x}(k+1) = G\hat{x}(k) + Hu(k) + K_e\left[y(k) - \hat{y}(k)\right]$$

$$= \left(G - K_eC\right)\hat{x}(k) + Hu(k) + K_eCx(k)$$

First, we define

$$Q = \left(WN^T\right)^{-1}$$

Where

$$N = \left[C^T \vdots \ \ G^TC^T \vdots \ \ \cdots \ \ \cdots \ \ \left(G^T\right)^{n-1}C^T\right]$$

and

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & & \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Where $a_1, a_2, \ldots, a_{n-1}$ are coefficients in the characteristic equation of the original state equation given by

$$|ZI - G| = Z^n + a_1 Z^{n-1} + \ldots + a_{n-1} Z + a_n = 0$$

Next, define

$$x(k) = Q\xi(k)$$

$$\xi(k+1) = Q^{-1}G\xi(k) + Q^{-1}Hu(k)$$

$$y(k) = CQ\xi(k)$$

Where

$$Q^{-1}GQ = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & & 1 & -a_1 \end{bmatrix}$$

$$CQ = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Now define

$$\hat{x}(k) = Q\hat{\xi}(k)$$

$$\hat{\xi}(k+1) = Q^{-1}(G - K_e C)Q\hat{\xi}(k) + Q^{-1}Hu(k) + Q^{-1}K_e CQ\xi(k)$$

$$\xi(k+1) - \hat{\xi}(k+1) = Q^{-1}GQ\xi(k) - Q^{-1}GQ\hat{\xi}(k) + Q^{-1}K_e CQ\hat{\xi}(k) - Q^{-1}K_e CQ\xi(k)$$

$$= Q^{-1}GQ\left[\xi(k) - \hat{\xi}(k)\right] + Q^{-1}K_e CQ\left[\hat{\xi}(k) - \xi(k)\right]$$

$$= \left[Q^{-1}GQ - Q^{-1}k_e CQ\right]\left[\hat{\xi}(k) - \xi(k)\right]$$

Define

$$e(k) = \xi(k) - \hat{\xi}(k)$$

Then eqn. becomes

$$e(k+1) = Q^{-1}(G - K_e C)Qe(k)$$

We require the error dynamics to be stable and $e(k)$ to reach zero with sufficient speed. The procedure for determining matrix K_e is first to select the desired observer poles (the eigen values of$[G - K_e C]$) and then to determine matrix K_e so that it will give the desired poles. If we require $e(k)$ reaching zero as fast as possible, then we require the error response to be deadbeat, so we must select all eigen values of $[G - K_e C]$ to be zero.

$$Q = \left(WN^T\right)^{-1}$$

$$Q^{-1} = WN^T = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & & \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \\ CG^n \end{bmatrix}$$

$$Q^{-1}K_e = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & & \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \\ CG^n \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_{n-1} \\ K_n \end{bmatrix}$$

Where $K_e = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_{n-1} \\ K_n \end{bmatrix}$

Since $Q^{-1}K_e$ is an n-vector, let us write

$$Q^{-1}K_e = \begin{bmatrix} \delta_n \\ \delta_{n-1} \\ \vdots \\ \vdots \\ \delta_1 \end{bmatrix}$$

$$Q^{-1}K_eCQ = \begin{bmatrix} \delta_n \\ \delta_{n-1} \\ \vdots \\ \vdots \\ \delta_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \delta_n \\ 0 & 0 & \cdots & \cdots & \delta_{n-1} \\ \vdots & \vdots & & & \\ \vdots & \vdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \cdots & \delta_1 \end{bmatrix}$$

$$Q^{-1}(G - K_eC)Q = Q^{-1}GQ - Q^{-1}K_eCQ$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n - \delta_n \\ 0 & 0 & \cdots & \cdots & -a_{n-1} - \delta_{n-1} \\ \vdots & \vdots & & & -a_{n-2} - \delta_{n-2} \\ \vdots & \vdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \cdots & -a_1 - \delta_1 \end{bmatrix}$$

The characteristic equation of the system is $\left| ZI - Q^{-1}(G - K_eC)Q \right| = 0$

$$\begin{vmatrix} Z & 0 & 0 & 0 & \cdots & a_n + \delta_n \\ -1 & Z & 0 & 0 & \cdots & a_{n-1} + \delta_{n-1} \\ 0 & -1 & Z & \cdots & 0 & a_{n-2} + \delta_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & Z + a_1 + \delta_1 \end{vmatrix} = 0$$

$$Z^n + (a_1 + \delta_1)Z^{n-1} + \ldots + (a_n + \delta_n) = 0$$

Suppose that the desired characteristics for the error dynamics is

$$(Z - \mu_1)(Z - \mu_2)\cdots\cdots(Z - \mu_n) = 0$$

$$Z^n + \alpha_1 Z^{n-1} + \alpha_2 Z^{n-2} + \cdots\cdots + \alpha_{n-1}Z + \alpha_n = 0$$

Where μ_1 are the desired eigen values of the system.

By comparing eqns.

$$\alpha_1 = a_1 + \delta_1$$
$$\alpha_2 = a_2 + \delta_2$$
$$\vdots$$
$$\alpha_n = a_n + \delta_n$$

From the above equations we get

$$\alpha_1 - a_1 = \delta_1$$
$$\alpha_2 - a_2 = \delta_2$$
$$\vdots$$
$$\alpha_n - a_n = \delta_n$$

$$\therefore Q^{-1}K_e = \begin{bmatrix} \delta_n \\ \delta_{n-1} \\ \vdots \\ \vdots \\ \delta_1 \end{bmatrix} = \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix}$$

Hence

$$K_e = Q\begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix} = \left(WN^T\right)^{-1}\begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix}$$

Equation gives the necessary observer feedback gain matrix K_e

Problem-2: Consider the double integrator system given by the equation

$$x(k+1) = Gx(k) + Hu(k)$$

$$y(k) = Cx(k)$$

where $G = \begin{bmatrix} 0 & T \\ 0 & 1 \end{bmatrix}$ $H = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Design a state observer for this system. It is desired that the error vector exhibit deadbeat response.

Solution: Given that

$$G = \begin{bmatrix} 0 & T \\ 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$G^T H^T = \begin{bmatrix} 0 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ T \end{bmatrix}$$

Observability matrix $Q_0 = \begin{bmatrix} C^T & \vdots & G^T C^T \end{bmatrix} = \begin{bmatrix} 0 & T \\ 0 & 1 \end{bmatrix}$

$|Q_0| \neq 0$, hence rank=2=n. Thus the system is completely observable.

The characteristic equation of the system is

$$|ZI - G| = \begin{vmatrix} Z & 0 \\ 0 & Z \end{vmatrix} - \begin{bmatrix} 0 & T \\ 0 & 1 \end{bmatrix}$$

$$= \begin{vmatrix} Z-1 & -T \\ 0 & Z-1 \end{vmatrix} = Z^2 - 2Z + 1 = 0$$

$$\Rightarrow Z^2 - 2Z + 1 = 0$$

Comparing the characteristic equation with

$$Z^2 + a_1 Z + a_2 = 0$$

$$\Rightarrow a_1 = -2, a_2 = 1$$

Since the deadbeat response is desired, the desired characteristic equation for the error dynamics is $Z^2 + \alpha_1 Z + \alpha_2 = Z^2 = 0$

$$\Rightarrow \alpha_1 = 0, \alpha_2 = 0$$

<u>Method -1</u>

The feedback gain matrix K_e is

$$K_e = Q \begin{bmatrix} \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix} = \left(WN^T\right)^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$N = \begin{bmatrix} C^T & \vdots & G^T C^T \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & T \end{bmatrix}$$

$$W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$WN^T = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix} = \begin{bmatrix} -1 & T \\ 1 & 0 \end{bmatrix}$$

$$\left(WN^T\right)^{-1} = \begin{bmatrix} -1 & T \\ 1 & 0 \end{bmatrix}^{-1} = \frac{1}{-T} \begin{bmatrix} 0 & -T \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/T & 1/T \end{bmatrix}$$

$$K_e = \left(WN^T\right)^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/T & 1/T \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1/T \end{bmatrix}$$

Method-2

The feedback gain matrix K_e according to Ackermann's formula

$$K_e = \Phi(G) \begin{bmatrix} C \\ CG \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Where

$$\Phi(G) = G^2 + \alpha_1 G + \alpha_2 I = G^2$$

$$= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2T \\ 0 & 1 \end{bmatrix}$$

Solution: Given that

$$G = A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad H = B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = D = \begin{bmatrix} 2 & 0 \end{bmatrix}$$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0$

The feedback gain matrix K_e is

$$K_e = Q\begin{bmatrix} \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix} = \left(WN^T\right)^{-1}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$N = \left[C^T \;\; \vdots \;\; G^T C^T\right] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$WN^T = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & 0 \end{bmatrix}$$

$$\left(WN^T\right)^{-1} = \begin{bmatrix} -2 & 2 \\ 2 & 0 \end{bmatrix}^{-1} \frac{1}{-4}\begin{bmatrix} 0 & -2 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$K_e = \left(WN^T\right)^{-1}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$