UNIT-1: INTRODUCTION TO SIGNALS

INTRODUCTION

The analysis of linear control system is based on the fact that the signals at various points in the system are continuous with respect to time. However, in some applications it is convenient to use any one or more control signals at discrete time intervals of time, for example in some industrial process control applications the signal is available only in sampled data form as a sequence of pulses. The control system using one or more signals at discrete time intervals are known as sampled data or digital or discrete time control systems. Digital controllers are used for achieving optimal performance-for example, in the form of maximum productivity, maximum profit, minimum cost or minimum energy use.

Generally the controllers are used in control system to modify the error signal for better control action. The controllers are classified into two types

1. Analog controllers
2. Digital controllers

Analog controllers:

- These are constructed using analog elements and their i/p and o/p are analog signals, which are continuous function of time.
- Complex, costlier and once fabricated, it is difficult to alter the controllers.

Digital controllers:

- These are constructed using non-programmable devices, microprocessor based systems or computer based systems.
- These are used complex and time shared control functions.
- Simple, versatile, programmable, fast acting and less costlier.
- It is easy to alter the control functions by modifying the program instructions.

CONTINUOUS TIME VERSUS DISCRETE TIME CONTROL SYSTEMS

If all the system variables of a control system are functions of time, it is termed as a continuous time control system

Ex: The speed control of a d.c motor with tacho-generator feedback.

If one or more system variables of control system are known at a certain discrete time, it is timed as a discrete time control system.

Ex: The microprocessor or computer based system.

The i/p and o/p signals of discrete time systems are digital or discrete, but the i/p and o/p signals of continuous time systems are analog or continuous time signals.

Continuous time systems, whose signals are continuous in time, may be described by differential equations where as in discrete time systems the signals are digital signal or sampled data signals may be described by difference equations.

BLOCK-DIAGRAM OF A DIGITAL CONTROL SYSTEM

A control system which uses a digital computer as a controller or compensator is known as digital control system. The advantages of using a digital computer for compensation include:accuracy, reliability, economy and most importantly, flexibility.

A block diagram of a digital control system is shown in the fig.. The basic elements of the system are shown by the blocks. The controller operation is controlled by clock. The input and output signal in a digital computer will be will be digital signal, but the error signal (input to the controller) to be modified by the controller and the control signal (output of the controller) to drive the plant are analog in nature. Hence a sample and hold circuit and an analog to digital converter (ADC) are provided at the digital computer input. A digital to analog converter (DAC) and a hold circuit are provided at the digital computer output.

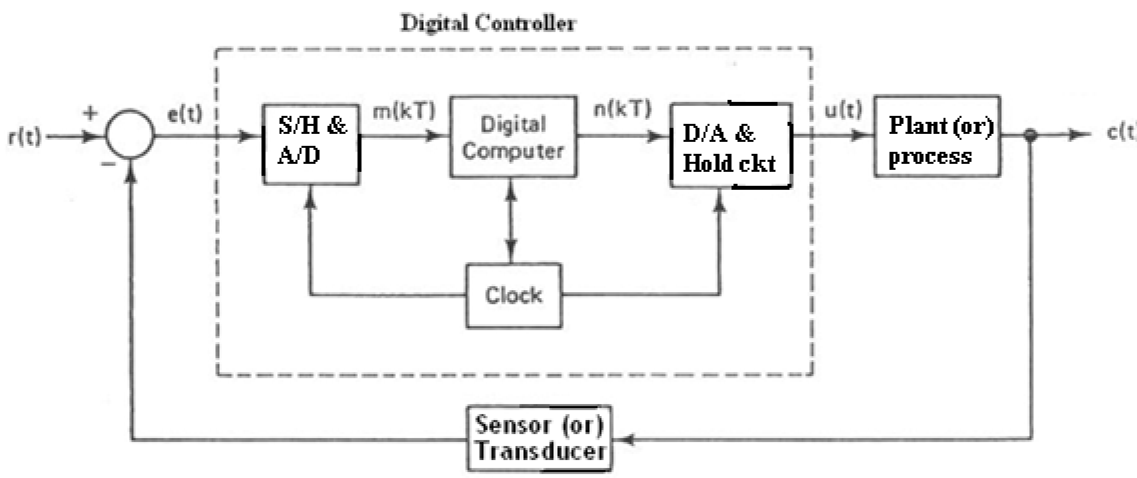


Fig.: Block-diagram of a digital control system

The sampler (S/H circuit) converts the continuous time error signal into a sequence of pulses and ADC produces binary code (binary number) of each sample. These codes are input data to the digital computer which process the binary code by means of an algorithm and produces another stream of binary codes as output. The DAC and hold circuit converts the output binary code to continuous time signal (analog signal), called control signal is fed to the plant, either directly or through an actuator to drive the plant (or to control its dynamics).

The operation that transforms continuous time signals into discrete-time data is called sampling or discretization or encoding. The inverse operation, the operation that transform discrete time data into a continuous time signal is called data hold or decoding; it amounts to a reconstruction of a continuous time signal from the sequence of discrete time data. The function of each block in the block diagram is given below

Sample and hold circuit: Sample and hold circuit is a general term used for sample and hold amplifier. It describes a circuit that receives an analog input signal and holds this signal at a constant value for a specified period of time. Usually the signal is electrical, but other forms are possible such as optical and mechanical.

Analog to digital converter (ADC): An analog to digital converter, also called an encoder, is a device that converts an analog signal into a digital signal, usually a numerically coded signal. Such a converter is needed as an interface between an analog component and a digital component. A sample and hold circuit is often an integral part of a commercially available A/D converter.

The operation of A/D conversion can be explained by the following block diagrams. The A/D conversion operation is carried out in three stages: sample and hold, quantization and encoding.

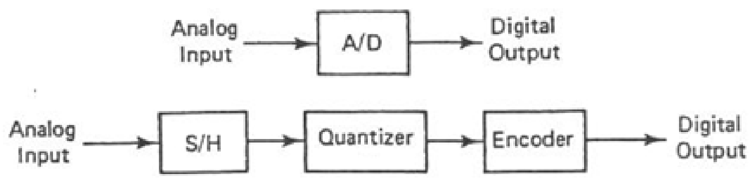


Fig: A/D Conversion

Digital to analog converter (DAC): A digital to analog converter, also called as a decoder, is a device that converts a digital signal into an analog signal. Such a converter is needed as an interface between a digital component and an analog component.

The operation of D/A conversion can be explained by the following block diagrams. Two stages are involved in the D/A conversion process: decoding and holding.

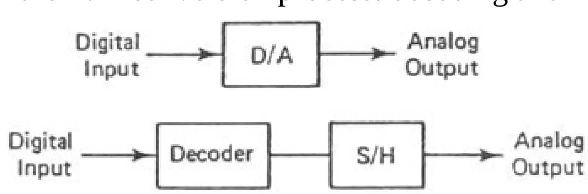


Fig: D/A Conversion

Plant or process: A plant is any physical object to be controlled, such as a furnace, a chemical reactor and a set of mechanical parts functioning together to perform a particular operation, such as a servo system or a space craft.

A process is generally defined as a progressive operation or development marked by a series of gradual changes that succeed one another in a relatively fixed way and lead towards a particular result or end.

Transducer: A transducer is a device that converts an input signal into an output signal of another form, such as a device that converts a pressure signal into a voltage output. The output signal in general, depends on past history of the input. Transducers may be classified as analog transducers and sampled-data transducers or digital transducers.

ADVANTAGES & DISADVANTAGES OF DIGITAL CONTROL SYSTEM

Advantages of Digital Control System:

The advantages of digital control system are listed below:

- Digital components are less susceptible to ageing and environmental variations.
- They are less sensitive to noise and disturbance.
- Digital processors are more compact and light weight.
- They are highly accurate, fast and flexible and more reliable.
- They are growing cheaper in cost.
- Provide high sensitivity to parameter variations.
- Allow more flexibility in programming without an alternation in hardware.
- Digital coded signals can be stored, transmitted, retransmitted, detected, analyzed or processed as desired.
- They are more reliable.

Disadvantages of Digital Control System:

Some of the disadvantages of digital control systems are as follows:

- Conversion of analog signals into discrete time signals and reconstruction introduces noise and errors in the signal
- Limitations on computing speed and signal resolution.
- Time delays caused in the control loops due to the limitation on computing speed.
- System instability as limit cycles in the closed loop due the finite word length of the processor.

Time Shifting

Suppose that we have a signal $x(t)$ and we define a new signal by adding/subtracting a finite time value to/from it. We now have a new signal, $y(t)$. The mathematical expression for this would be $x(t \pm t_0)$. Graphically, this kind of signal operation results in a positive or negative “shift” of the signal along its time axis. However, note that while doing so, none of its characteristics are altered. This means that the time-shifting operation results in the change of just the positioning of the signal without affecting its amplitude or span.

Let's consider the examples of the signals in the following figures in order to gain better insight into the above information.

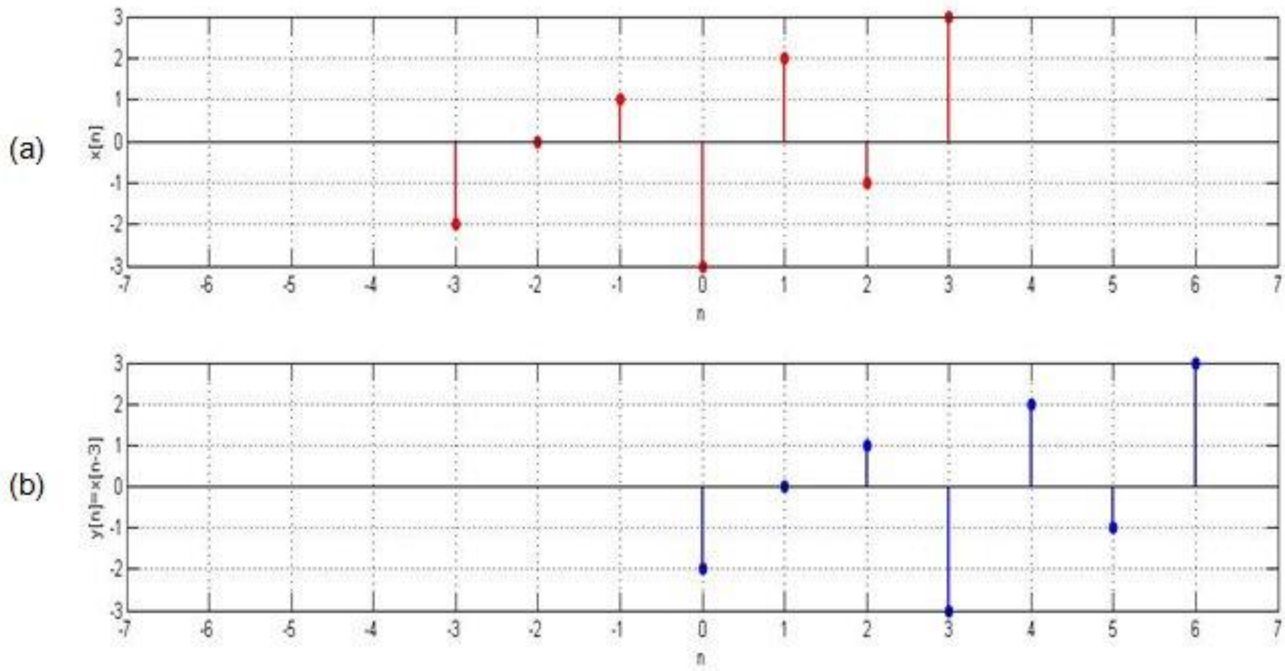


Figure 1. Original signal and its time-delayed version

Here the original signal, $x[n]$, spans from n = -3 to n = 3 and has the values -2, 0, 1, -3, 2, -1, and 3, as shown in Figure 1(a).

Time-Delayed Signals

Suppose that we want to move this signal right by three units (i.e., we want a new signal whose amplitudes are the same but are shifted right three times).

This means that we desire our output signal $y[n]$ to span from n = 0to n = 6. Such a signal is shown as Figure 1(b) and can be mathematically written as $y[n] = x[n-3]$.

This kind of signal is referred to as time-delayed because we have made the signal arrive three units late.

Time-Advanced Signals

On the other hand, let's say that we want the same signal to arrive early. Consider a case where we want our output signal to be advanced by, say, two units. This objective can be accomplished by shifting the signal to the left by two time units, i.e., $y[n] = x[n+2]$.

The corresponding input and output signals are shown in Figure 2(a) and 2(b), respectively. Our output signal has the same values as the original signal but spans from n = -5 to n = 1 instead of n = -3 to n = 3. The signal shown in Figure 2(b) is aptly referred to as a time-advanced signal.

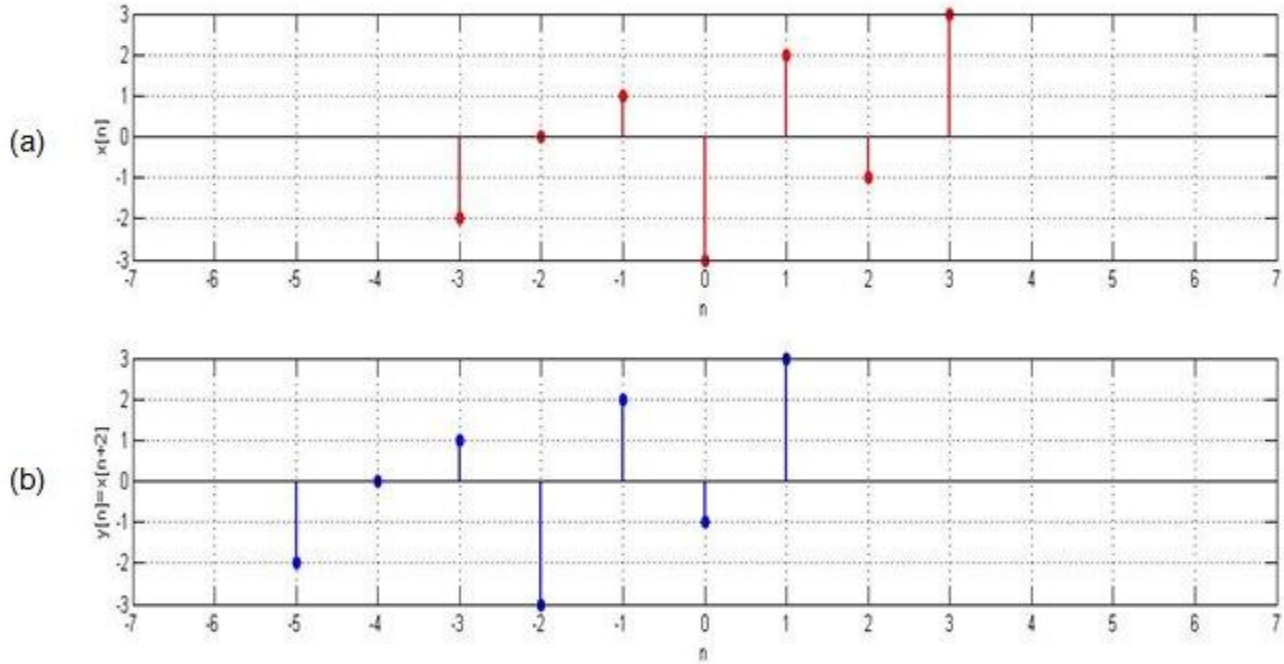


Figure 2. Original signal and its time-advanced version

For both of the above examples, note that the time-shifting operation performed over the signals affects not the amplitudes themselves but rather the amplitudes with respect to the time axis. We have used discrete-time signals in these examples, but the same applies to continuous-time signals.

Time Scaling

Now that we understand more about performing addition and subtraction on the independent variable representing the signal, we'll move on to multiplication.

For this, let's consider our input signal to be a continuous-time signal $x(t)$ as shown by the red curve in Figure 3.

Now suppose that we multiply the independent variable (t) by a number greater than one. That is, let's make t in the signal into, say, $2t$. The resultant signal will be the one shown by the blue curve in Figure 3.

From the figure, it's clear that the time-scaled signal is contracted with respect to the original one. For example, we can see that the value of the original signal present at $t = -3$ is present at $t = -1.5$ and those at $t = -2$ and at $t = -1$ are found at $t = -1$ and at $t = -0.5$ (shown by green dotted-line curved arrows in the figure).

This means that, if we multiply the time variable by a factor of 2, then we will get our output signal contracted by a factor of 2 along the time axis. Thus, it can be concluded that the multiplication of the signal by a factor of *n* leads to the compression of the signal by an equivalent factor.Now, does this mean that dividing the variable *t* by a number greater than 1 will cause the signal to become expanded? That is, if we divide the variable *t* by a factor of *n*, will we get a signal which is stretched by an equivalent factor?

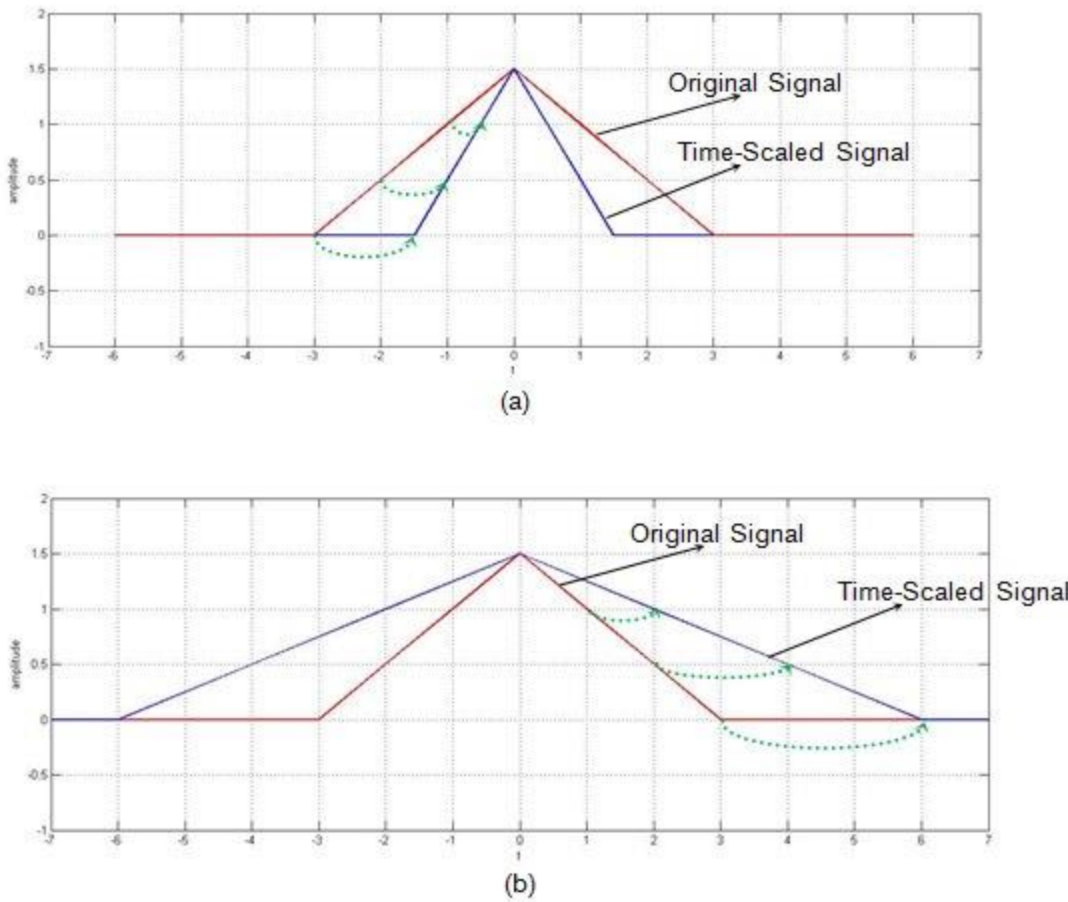


Figure 3. Original signal with its time-scaled versions

For this, let's consider our signal to be the same as the one in Figure 3 (the red curve in the figure). Now let's multiply its time-variable *t* by ½ instead of 2. The resultant signal is shown by the blue curve in Figure 3(b). You can see that, in this time-scaled signal indicated by the green dotted-line arrows in Figure 3(b), we have the values of the original signal present at the time instants $t = 1, 2$, and 3 to be found at $t = 2, 4$, and 6.

This means that our time-scaled signal is a stretched-by-a-factor-of-*n* version of the original signal. So the answer to the question posed above is "yes."

Although we have analyzed the time-scaling operation with respect to a continuous-time signal, this information applies to discrete-time signals as well. However, in the case of discrete-time signals, time-scaling operations are manifested in the form of decimation and interpolation.

Periodic and Non-periodic Signals

A continuous-time periodic signal exhibits the same value at integer multiples of T , where T is called the period of the signal.

Analogously, a discrete-time periodic signal does not change its value for any discrete-time shift N. Mathematically, for any t and N we have, respectively,

$$x(t) = x(t + T) \text{ and}$$

$$x[n] = x[n + N].$$

When above equation is not satisfied for a continuous-time and a discrete-time signal, respectively, we say that the signal is non-periodic (or aperiodic).

Linear time invariant and causal systems

Linear and Non Linear systems:

A system is said to be linear if it follows both the Homogenity and superposition principles.

Homogenity: If the input is multiplied by a constant, the output shall also be multiplied by the same.

Superposition: If the input is superposed by two signals, the out put shall also be superposed.

So, a general description of a linear system is

If X1,2(n) -> Y1,2(n) => aX1(n)+bX2(n) => aY1(n)+bY2(n)

Anything, which is not a linear system, which means that it doesn’t follow either of the above properties or all of them, the system is called non linear.

Time Invariant and Variant Systems:

In digital signal processing, we can easily observe that time has lost its significance. So this is also called shift invariance. If the output waveform is preserved even after shifting the signal by a period of N and the body of the waveform is exactly preserved, this is called a Time Invariant system. Any system which do not follow the above specification is a time variant system. A general description of time invariant system is

X(n) à Y(n) => X(n-N) à Y(n-N)

A system which obeys both the linearity and time invariance are called linear time invariant systems, abbreviated as LTI systems.

Causality

An LTI system is *causal* if its output *y*(*t*) only depends on the current and past input *x*(*t*) (but not the future). Assuming the system is initially at rest, i.e., its output is *y*(*t*)=0, then if $x(t) = \delta(t)$ which occurs at the moment *t*=0, the output *y*(*t*)=*h*(*t*) will be none zero only when $t \geq 0$, i.e., *h*(*t*)=0 for *t*<0. But as the output and input of an LTI is related by convolution, we have

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

Moreover, if the input begins at a specific moment, e.g., *x*(*t*)=0 for *t*<0, then we have

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{t} h(\tau)x(t-\tau)d\tau$$

THE Z-TRANSFORM

The z-transform method is an operational method that is very powerful when working with discrete time systems. For a given sequence values of x(kT), its z-transform is defined by X(z) and is given by

$$X(z)=Z[x(kT)]=\sum_{k=-\infty}^{k=+\infty} x(kT)z^{-k}$$

Where Z is a complex variable.

The sequence of the above equation is considered to be two sided and the transform is called two-sided z-transform, since the time index k is defined for both positive and negative values. If the sequence x(kT) is one sided sequence [i.e x(kT) is defined only for the positive values of k] than the z-transform is called one sided z-transform.

The one sided z-transform of the X(kT) is defined as

$$X(z) = Z[x(kT)] = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

Z-TRANSFORMS OF SOME STANDARD FUNCTIONS

i. Unit step function: Consider the unit step function

$$x(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$X(z) = Z[1] = \sum_{K=0}^{\infty} 1 \times z^{-K}$$

$$= 1 + z^{-1} + z^{-2} + \ldots\ldots\ldots\ldots$$

$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

i.e. if x(t)=1, then X(z)=$\frac{z}{z-1}$

ii. Unit ramp function: Consider the unit ramp function

$$x(t)=\begin{cases} t, & t\geq 0 \\ 0, & t<0 \end{cases}$$

Put t=kT, k= 0, 1, 2,………

$$X[z]=Z[kT]=\sum_{k=0}^{\infty}(kT)z^{-k}$$

$$=\mathrm{T}\sum_{k=0}^{\infty}kz^{-k}$$

$$=\mathrm{T}[z^{-1}+2z^{-2}+3z^{-3}+\dots\dots\dots\dots]$$

We know that

$$X(z)=Z[1]=[1+z^{-1}+z^{-2}+\dots\dots\dots\dots]=\frac{z}{z-1}$$

Differentiating both sides w.r.t 'z'

$$-z^{-2}-2z^{-3}-3z^{-4}-\dots\dots\dots\dots=\frac{(z+1)-z}{(z-1)^2}=-\frac{1}{(z-1)^2}$$

$$\Rightarrow z^{-1}+2z^{-2}+3z^{-3}+\dots\dots\dots\dots=\frac{z}{(z-1)^2}$$

From eqns.(2.3) and (2.4), we have

$$X(z)=T\times\left[\frac{z}{(z-1)^2}\right]=\frac{zT}{(z-1)^2}$$

i.e., if $x(kT)=kT$, then $X(z)=\frac{zT}{(z-1)^2}$

iii. Exponential function: Consider the exponential function

$$x(t)=\begin{cases} e^{-at}, & t\geq 0 \\ 0, & t<0 \end{cases}$$

Put t=kT, k= 0, 1, 2,………

Put t=kT, k= 0, 1, 2,………

$x(kT) = e^{-akT}$, k=0,1,2……..

$$X(z) = Z[e^{at}] = \sum_{k=0}^{\infty} x(kT) z^{-k}$$

$$= \sum_{k=0}^{\infty} e^{-akT} z^{-k}$$

$$= 1 + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + e^{-3aT} z^{-3} + \ldots\ldots\ldots$$

$$= 1 + [e^{-at} z^{-1}]^1 + [e^{-at} z^{-1}]^2 + \ldots\ldots\ldots\ldots\ldots$$

$$= \frac{1}{1 - e^{-aT} z^{-1}}$$

$$= \frac{z}{z - e^{-aT}}$$

i.e., if $x(t) = e^{-at}$ then $X(z) = \dfrac{z}{z - e^{-aT}}$

<u>Note:</u> Sum of the infinite series, $1 + +r^2 + r^3 + r^4 + \ldots\ldots = \dfrac{1}{1-r}$

iv. Polynomial function, a ^{t} : Let us obtain the z-transform of x(t) as defined by

$$x(t) = \begin{cases} a^t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Put t=kT, k= 0, 1, 2,………

$x(kT) = \; a^{kT}$

$$X(z) = Z[a^{kT}] = \sum_{k=0}^{\infty} a^{kT} z^{-k}$$

$$= 1 + a^T z^{-1} + a^{2T} z^{-2} + a^{3T} z^{-3} + ...$$

$$= 1 + (a^T z^{-1})^1 + (a^T z^{-1})^2 + ...$$

$$= \frac{1}{1 - a^T Z^{-1}} = \frac{z}{z - a^T}$$

i.e., if $x(t) = a^t$, then $X(z) = \dfrac{z}{z - a^T}$

v) Sinusoidal function: Consider the sinusoidal function

$$x(t) = \begin{cases} \sin \omega t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Noting that

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$
$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

$x(kT) = \sin \omega kT$ for k=0,1,2.....

$$X(z) = Z[\sin \omega t] = Z[\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})]$$

$$= \frac{1}{2j}[\frac{z}{z - e^{j\omega t}} - \frac{z}{z - e^{-j\omega t}}]$$

$$= \frac{z}{2j}[\frac{z - e^{-j\omega t} - z + e^{j\omega t}}{z^2 - z(e^{j\omega t} + e^{-j\omega t}) + 1}$$

$$= \frac{z}{2j} \times \frac{2j \sin \omega t}{z^2 - 2z \cos \omega t + 1}$$

$$= \frac{z \sin \omega t}{z^2 - 2z \cos \omega t + 1}$$

If $x(t) = \sin \omega t$, then $X(z) = \dfrac{z \sin \omega t}{z^2 - 2z \cos \omega t + 1}$

we have,

$$\sin \omega t = \frac{1}{2j}[e^{j\omega t} - e^{-j\omega t}]$$

$$\cos \omega t = \frac{1}{2}[e^{j\omega t} + e^{-j\omega t}]$$

Put t=kT, k=0,1,2…..

vi) Cosine function: consider the cosine function

$$x(t) = \begin{cases} \cos \omega t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Put t=kT, k=0,1,2…..

$x(kT) = \cos \omega kT$, k=0,1,2,3.....

$$X(z) = Z[\cos \omega t] = Z\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right] = \frac{1}{2}[\frac{z}{z - e^{j\omega t}} + \frac{z}{z + e^{-j\omega t}}]$$

$$= \frac{z}{2}[\frac{z - e^{-j\omega t} + z - e^{j\omega t}}{z^2 - z(e^{j\omega t} + e^{-j\omega t}) + 1}]$$

$$= \frac{z}{2}[\frac{2z - e^{j\omega t} - e^{-j\omega t}}{z^2 - 2z\cos \omega t + 1}]$$

$$= \frac{z(z - \cos \omega t)}{z^2 - 2z\cos \omega t + 1}$$

i.e. if $x(t) = \cos \omega t$, then $X(z) = \dfrac{z(z - \cos \omega t)}{z^2 - 2z \cos \omega t + 1}$

vii) x(k) = k^P, P being a positive integer

$$X(z) = Z[k^P]$$

$$= \sum_{k=0}^{\infty} k^P z^{-k}$$

Also by definition

$$Z(k^{P-1}) = \sum_{k=0}^{\infty} k^{P-1} z^{-k}$$

Differentiating of eqn.(2.6) with respect to 'z'

$$\frac{d}{dz}\{Z[k^{P-1}]\} = \sum_{k=0}^{\infty} k^{P-1}(-k) z^{-(k+1)}$$

From eqns. (2.5) and (2.7)

$$Z(k^P] = -z\frac{d}{dz}\{Z[k^{P-1}]\}$$

This gives a recurrence formula.

Example: $Z[k] = \dfrac{Z}{(Z-1)^2}$

We know that

Problem-1: Obtain the Z-transforms of X(s)= $\frac{1}{S(S+1)}$

Solution:

$$x(t) = L^{-1}[X(s)] = L^{-1}\left[\frac{1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$= 1 - e^{-t}$$

Put t=kT, $x(kT) = 1 - e^{-kT}$

$$X(z) = Z[1 - e^{-kT}]$$

$$= Z[1] - Z[e^{-kT}]$$

$$= \frac{z}{z-1} - \frac{z}{z-e^{-T}}$$

$$= \frac{z(z - e^{-T} - z + 1)}{(z-1)(z-e^{-T})}$$

$$= \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}$$

Problem-2: Obtain the z-transform of k^2?

Solution:

$$Z[k^2] = \sum_{k=0}^{\infty} k^2 Z^{-k}$$

$$Z[k^P] = -Z\frac{d}{dz}\{Z[k^{P-1}]\}$$

Put P=1

$$Z[k] = -Z\frac{d}{dz}\{Z[1]\}$$

$$= -Z\frac{d}{dz}(\frac{Z}{Z-1})$$

$$= -Z[\frac{(Z-1)\times 1 - Z\times 1}{(Z-1)^2} \quad = \frac{Z}{(Z-1)^2}$$

$$\Rightarrow Z[k] = \frac{Z}{(Z-1)^2}$$

$$Z[k^P] = -z\frac{d}{dz}\{Z[k^{P-1}]\}$$

Put P=2,

$$Z[k^2] = -z\frac{d}{dz}\{Z[k]\} = -z\frac{d}{dz}[\frac{z}{(z-1)^2}]$$

$$= -z[\frac{(z-1)^2 \times 1 - z \times 2 \times (z-1)}{(z-1)^4}]$$

$$= -z\frac{(1-z^2)}{(z-1)^4} = \frac{-z(1-z)(1+z)}{(z-1)^4} = \frac{z(z+1)}{(z-1)^3}$$

Problem-3: Obtain the z-transform of $\frac{1}{(k+1)!}$?

Solution:

$$Z\left[\frac{1}{(k+1)!}\right]] = \sum_{k=0}^{\infty}\frac{1}{(k+1)!}z^{-k}$$

$$= \sum_{k=0}^{\infty}\frac{1}{(k+1)!z^k}$$

$$= 1 + \frac{1}{2!z} + \frac{1}{3!z^2} + \frac{1}{4!z^3} + \ldots\ldots\ldots$$

$$= z[\frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{z^3 3!} + \ldots\ldots\ldots\ldots]$$

$$= z[1 + \frac{\frac{1}{z}}{1!} + \frac{\frac{1}{z^2}}{2!} + \frac{\frac{1}{z^3}}{3!} + \ldots\ldots\ldots] - z$$

$$= ze^{\frac{1}{z}} - z \quad = z(e^{\frac{1}{z}} - 1)$$

<u>**Note:**</u> $e^x = 1 + \frac{x}{1!} + \frac{x}{2!} + - - - - -$

IMPORTANT PROPERTIES AND THEOREMS OF THE Z-TRANSFORM

The use of the z-transform method in the analysis of discrete time control systems is very helpful if the theorems of the z-transforms are discussed. In this section we present important properties and useful theorems of the z-transforms. We assume that the time function x(t) is z-transformable and x(t) is zero for t<0.

(i) Multiplication by a constant: If x(z) is the z-transform of x(t), then

Z[(ax(t)] = aZ[x(t)] = aX(z), Where 'a' is a constant

Proof: $Z[ax(t)] = \sum_{k=0}^{\infty} ax(KT)z^{-K} = a\sum_{k=0}^{\infty} x(KT)z^{-K} = aX(z)$

$$\therefore Z[ax(t)] = aX(z)$$

(ii) Linearity of the z-transform: If f(k) and g(k) are z-transformable and a and b are constants, then x(k) is formed by a linear combination, x(k) = af(k) + bg(k) has the z-transform X(z) = aF(z) + bG(z). Where F(z) and G(z) are the z-transforms of f(x) and g(k) respectively.

Proof: $X(z) = z[x(k)] = z[af(k) + bg(k)] = \sum_{k=0}^{\infty}[af(k) + bg(k)]z^{-k}$

$$= a\sum_{k=0}^{\infty} f(K)z^{-k} + b\sum_{k=0}^{\infty} g(K)z^{-k}$$

$$= aZ[f(k)] + bZ[g(k)] = aF(z) + bG(z)$$

$$\therefore Z[af(k) + bg(k)] = aF(z) + bG(z)$$

(iii) Multiplication by a^k: If X(z) is the z-transform of x(k), then the z-transform of $a^k x(k)$ can be given by $X(a^{-1}z)$

i.e. $Z[a^k x(k)] = X(a^{-1}z) = X(z/a)$

Proof: $Z[(a^k x(k)] = \sum_{k=0}^{\infty} a^k x(k) z^{-k}$

$$= \sum_{K=0}^{\infty} x(K)(a^{-1}z)^{-K} = X(a^{-1}z) = X(z/a)$$

$$\therefore Z[a^k x(k)] = X(z/a)$$

(iv) Shifting theorem or real translation theorem: If x(t) = 0 for t<0 and x(t) has the transform X(z), then

(a) $z[x(t-nT)] = z^{-n} X(z)$

(b) $z[x(t+nT)] = z^{n}[X(z) - \sum_{K=0}^{n-1} x(kT) z^{-K}$

Proof:

(a) $Z[x(t-nT)] = \sum_{K=0}^{\infty} x(KT-nT) z^{-K}$

$$= \sum_{k=0}^{\infty} x(k-n)Tz^{-k+n-n}$$

$$= z^{-n} \sum_{k=0}^{\infty} x[(k-n)T) z^{-(k-n)}$$

Put k-n = m

If k=0, m=-n

If k=∞, m=∞

$$\therefore Z[x(t-nT)] = z^{-n}\sum_{m=-n}^{\infty} x(mT)z^{-m}$$

Since x(mT) = 0 for m<0, we may change the lower limit on the summation from m=-n to m=0.

Hence

$$Z[x(t-nT)] = z^{-n}\sum_{m=0}^{\infty} x(mT)z^{-m} = z^{-n}X(z)$$

$$\therefore Z[x(t-nT)] = z^{-n}X(z)$$

(b) $Z[x(t+nT)] = \sum_{k=0}^{\infty} x(kT+nT)z^{-k}$

$$= \sum_{k=0}^{\infty} x(k+n)T\ z^{-k-n+n}$$

$$= z^{n}\sum_{k=0}^{\infty} x[(k+n)T)z^{-(k+n)}$$

Put m=k+n

If k=0, m=n

If k=∞, m=∞

$$\therefore Z[x(t+nT)] = z^n \sum_{m=n}^{\infty} x(mT) z^{-m}$$

$$= z^n \left[\sum_{m=n}^{\infty} x(mT) z^{-m} + \sum_{m=0}^{n-1} x(mT) z^{-m} - \sum_{m=0}^{n-1} x(mT) z^{-m} \right]$$

$$= z^n \left[\sum_{m=0}^{\infty} x(mT) z^{-m} - \sum_{m=0}^{n-1} x(mT) z^{-m} \right]$$

$$= z^n \left[X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k} \right]$$

$$\therefore Z[x(t+nT)] = z^n [X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k}$$

From the above equation, we obtain

$$Z[x(k+1)] = zX(z) - zx(0)$$

$$Z[x(k+z)] = z.Z[x(k+1)] - zx(1)$$

$$= z^2 X(z) - z^2 x(0) - zx(1)$$

Similarly,

$$Z[x(k+n)] = z^n X(z) - z^n x(0) - z^{n-1} x(1) - - - - - zx(n-1)$$

Discrete Function	**Z-Transform**
$x(k+4)$	$z^4 X(z) - z^4 x(0) - z^3 x(1) - z^2 x(2) - zx(3)$
$x(k+3)$	$z^3 X(z) - z^3 x(0) - z^2 x(1) - zx(2)$
$x(k+2)$	$z^2 X(z) - z^2 x(0) - zx(1)$
$x(k+1)$	$zX(z) - zx(0)$
$x(k)$	$X(z)$
$x(k-1)$	$z^{-1} X(z)$
$x(k-2)$	$z^{-2} X(z)$
$x(k-3)$	$z^{-3} X(z)$
$x(k-4)$	$z^{-4} X(z)$

Z-Transform of x(k+m) and (k-m)

(v)Complex translation theorem: If x(t) has the z-transform X(z), then the z-transform of x(t) can be given by $X(ze^{\pm aT})$

Proof: Pput t = kT

$$Z[e^{-at}x(t)] = Z[e^{-aKT}x(KT)]$$

$$= \sum_{k=0}^{\infty} e^{-akT}x(kT)z^{-k}$$

$$= \sum_{k=0}^{\infty} x(kT)(ze^{aT})^{-k} = X(ze^{aT})$$

$$\therefore Z[e^{\mp at}x(t)] = X(ze^{\pm aT})$$

Thus, we see that replacing z in X(z) by ze^{aT} gives the z-transform of $e^{-at}x(t)$

Problem-4: Obtain the z-transform of te^{-at}

Solution: we know that

$$Z[t] = \frac{zT}{(z-1)^2} = X(z)$$

According to complex translation theorem

$$Z[te^{-aT}] = X(ze^{aT}) = \frac{ze^{aT}T}{(ze^{aT}-1)^2} = T\frac{ze^{aT}}{(ze^{at}-1)^2}$$

Problem-5: Given the z-transforms of $\sin\omega t$ and $\cos\omega t$, obtain the z-transform of $e^{-at}\sin\omega t$ and $e^{-aT}\cos\omega t$ respectively by using the complex translation theorem.

Solution: We know that

$$Z[\sin\omega t] = \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1} = X(z)$$

$$Z[\sin \omega t] = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1} = X(z)$$

$$\therefore Z[e^{-at} \sin \omega t] = X(ze^{aT}) = \frac{ze^{aT} \sin \omega T}{z^2 - 2ze^{aT} \cos \omega T + 1}$$

$$Z[\cos \omega t] = \frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1} = X(z)$$

$$\therefore Z[e^{-at} \cos \omega t] = X(ze^{aT}) = \frac{ze^{aT}(ze^{aT} - \cos \omega T)}{z^2 e^{aT} - 2ze^{aT} \cos \omega T + 1}$$

Problem-6: Find the z-transform of the sequence

$$x(k) = \left(\frac{1}{2}\right)^k \text{ ,for k=0, 1, 2, 3 . . . . .}$$

Solution: $X(z) = Z[x(k)] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^{-k}$

$$= 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + -----$$

$$= 1 + \frac{1}{2z} + \left(\frac{1}{2z}\right)^2 + \left(\frac{1}{2z}\right)^3 + -----$$

$$= \left(\frac{1}{1 - \left(\frac{1}{2z}\right)}\right) = \frac{z}{z - 0.5}$$

Problem -7: Obtain the z-transform of

$$f(a) = \begin{cases} a^{k-1t}, & k = 1, 2, 3....... \\ 0, & k \le 0 \end{cases}$$

Solution: Let $x(k)=a^k$

$$Z[x(k)] = Z[a^k] = \frac{z}{z - a}$$

We know that

$$Z[x(k-1)] = Z[a^{k-1}] = z^{-1} X(z)$$

$$\therefore Z[f(a)] = Z[a^{k-1}] = z^{-1} \frac{z}{z-a} = \frac{1}{z-a} = \frac{z^{-1}}{1 - az^{-1}}$$

vi) Partial differentiation theorem: Let the z-transform of the function x (t,a) be represented by X(z, a), where 'a' is an independent variable or constant. The z-transform of the partial derivative of x (t,a) with respect to 'a' is given by

$$Z\left\{\frac{\partial}{\partial a}[x(t,a)]\right\} = \frac{\partial}{\partial a}[X(z,a)]$$

Proof: From the z-transform definition

$$Z\left\{\frac{\partial}{\partial a}[x(t,a)]\right\} = \sum_{k=0}^{\infty} \frac{\partial}{\partial a} x(kT,a) z^{-k}$$

$$= \frac{\partial}{\partial a} \sum_{k=0}^{\infty} x(kT,a) z^{-k}$$

$$= \frac{\partial}{\partial a} X(z,a)$$

$$\therefore Z\left\{\frac{\partial}{\partial a}[x(t,a)]\right\} = \frac{\partial}{\partial a} X(z,a)$$

Problem-9: Find the z-transform of f(t)= t e^{-at} by using partial differentiation theorem.

Solution: We know that

$$\frac{\partial}{\partial a}[-e^{-at}] = te^{-at}$$

$$Z[te^{-at}] = Z\left[\frac{\partial}{\partial a}(-e^{-at})\right]$$

$$= Z\left[-\frac{\partial}{\partial a}[e^{-at}]\right]$$

$$= -\frac{\partial}{\partial a} X(z,a) \text{ [}\because \text{By using partial Differentiation theorem]}$$

Where $X(z,a) = Z[e^{-at}] = \frac{z}{z - e^{-at}}$

Where $X(z,a) = Z[e^{-at}] = \frac{z}{z - e^{-at}}$

$$\therefore Z[te^{-at}] = -\frac{\partial}{\partial a}\left[\frac{z}{z - e^{-aT}}\right] = -\left[\frac{(z - e^{-aT}) \times 0 - ze^{-aT}(-T)}{(z - e^{-aT})^2}\right] = \frac{zTe^{-aT}}{(z - e^{-aT})^2}$$

Problem-10: Find the z-transform of f(t)= t^2 e^{-at} by using partial differentiation theorem.

Solution: We know that

$$\frac{\partial}{\partial a}\left[-te^{-at}\right] = t^2 e^{-at}$$

$$Z[t^2 e^{-at}] = Z\left[\frac{\partial}{\partial a}(-te^{-at})\right] = -Z\left[\frac{\partial}{\partial a} te^{-at}\right] = -\frac{\partial}{\partial a} X(z,a)$$

Where

$$X(z,a) = Z[te^{-at}] = \frac{zTe^{-aT}}{(z - e^{-aT})^2}$$

$$\therefore Z[t^2 e^{-at}] = -\frac{\partial}{\partial a}\left[\frac{Tze^{-aT}}{(z - e^{-aT})^2}\right]$$

$$= -TZ\left[\frac{\left(z-e^{aT}\right)^2 e^{-aT}(-T) - e^{-aT}(-T) - e^{-aT}2(z-e^{-aT})(-e^{-aT})(-T)}{\left[z-e^{-aT}\right]^4}\right]$$

$$= -zT(z-e^{-aT})\frac{[-Te^{-aT} - 2Te^{-2aT}]}{[z-e^{-aT}]^4}$$

$$= T^2 z\frac{[ze^{-aT} + e^{-2aT}]}{[Z-e^{-aT}]^3}$$

vii) Initial value theorem: If x(t) has the z-transform X(z) and $\underset{z\to\infty}{Lt}\,X(z)$ exists, then the initial value x (0) of x (t) or x (k) is given by

Proof: From the z-transform definition

$$Z[x(k)] = X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$$
$$= x(0) + x(1)z^{-1} + x(z)z^{-2} + -----$$

Applying both sides as $\underset{z\to\infty}{Lt}$

$$\underset{z\to\infty}{Lt}\,X(z) = \underset{z\to\infty}{Lt}\left[x(0) + \frac{x(1)}{z} + \frac{x(z)}{z^2} ----\right]$$
$$= x(0) + 0 + 0 + -----$$
$$= x(0)$$

$$\therefore \underset{k\to 0}{Lt}\,x(k) = x(0) = \underset{z\to\infty}{Lt}\,X(z)$$

viii) Final value theorem: If the function x(k) has the z-transform X(z) and if the function $(1-z^{-1})X(z)$ does not have poles on or outside the unit circle |z|=1 in the z-plane, then

$$\therefore \underset{k\to\infty}{Lt}\, x(k) = x(\infty) = \underset{z\to 1}{Lt}(1-z^{-1})X(z)$$

Proof: $Z[x(k)] = X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + --------$$

$$Z(x(k-1)) = z^{-1}X(z)$$

$$= \sum_{k=0}^{\infty} x(k-1)z^{-k}$$

By subtracting eqn.(2.19) from eqn.(2.18)

$$X(z) - z^{-1}X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k}$$

Applying both sides as $\underset{z\to 1}{Lt}$

$$\underset{z\to 1}{Lt}[X(z) - z^{-1}X(z)] = \underset{z\to 1}{Lt}[\sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k}]$$

$$= \sum_{k=0}^{\infty}[x(k) - x(k-1)]$$

$$= [x(0) - x(-1)] + [x(1) - x(0)] + [x(2) - x(1)] ------ + x(\infty)]$$

$$= x(\infty)$$

$$i.e.\ \underset{z\to 1}{Lt}[1-z^{-1}]X(z) = x(\infty) = \underset{k\to\infty}{Lt}\, x(k)$$

$$\therefore x(\infty) = \underset{k\to\infty}{Lt}\, x(k) = \underset{z\to 1}{Lt}(1-z^{-1})X(z)$$

Problem-11: Determine the initial value x(0) if the z-transforms of x(t) is given by

$$X(z) = \frac{(1-e^{-1})z^{-1}}{(1-z^{-1})(1-e^{-T}z^{-1})}$$

Solution: According to the initial value theorem

$$x(0) = \underset{z\to\infty}{Lt}\, X(z) = \underset{z\to\infty}{Lt}\left[\frac{(1-e^{-1})z^{-1}}{(1-z^{-1})(1-e^{-T}z^{-1})}\right] = 0$$

Problem-12: Determine the final value $x(\infty)$ of $x(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}, a > 0$ by using final value theorem?

Solution: According to final value theorem

$$x(\infty) = \underset{z\to 1}{Lt}(1-z^{-1})X(z)$$

$$= \underset{z\to 1}{Lt}(1-z^{-1})\left[\frac{1}{1-z^{-1}}-\frac{1}{1-e^{-aT}z^{-1}}\right]$$

$$= \underset{z\to 1}{Lt}\left[1-\frac{1-z^{-1}}{1-e^{-aT}z^{-1}}\right]=1$$

xi) Complex differentiation theorem

In the region of convergence a power series in z may be differentiated with respect to z any number of times to get a convergent series. The derivative of x(z) converge in the same region as X(z).

i.e., $Z[k^m x(k)] = \left(-z\frac{d}{dz}\right)^m X(z)$

Proof: Let us consider

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

Differentiating X(z), w.r.t z

$$\frac{d}{dz}X(z) = \sum_{k=0}^{\infty}(-k)x(k)z^{-k-1}$$

Multiplying both sides w.r.t (-z)

$$-z\frac{d}{dz}X(z) = \sum_{k=0}^{\infty}(k)x(k)z^{-k} = Z[kx(k)]$$

Differentiating eqn.(2.21), w.r.t 'z' on both sides

$$\frac{d}{dz}\left[-z\frac{d}{dz}X(z)\right] = \sum_{k=0}^{\infty}(-k^2)x(k)z^{-k-1}$$

$$-z\frac{d}{dz}\left[-z\frac{d}{dz}X(z)\right] = \sum_{k=0}^{\infty}k^2x(k)z^{-k} = Z[k^2x(k)]$$

$$\therefore \left(-z\frac{d}{dz}\right)^2 X(z) = Z[k^2x(k)$$

The operation $(-z\frac{d}{dz})^2$ implies that we apply the operator $-z\frac{d}{dz}$ twice.

Similarly, we can write in general

$$Z[k^m x(k)] = \left(-z\frac{d}{dz}\right)^m X(z)$$

Such a complex differentiation theorem enables us to obtain the new z-transform pairs from the known z-transform pairs.

Problem-15: Obtain the z-transform of unit ramp of x(k) with the knowledge of z transform of unit step signal.

Solution: For the unit step signal, $x(k) = 1$

$$\therefore X(z) = Z[1] = \frac{z}{z-1}$$

For the ramp signal, $x_1(k) = tx(k) = kT.1$

$$X_1(z) = Z[kT.1]$$

$$= -Tz\frac{d}{dz}X(z) = -Tz\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -Tz\left[\frac{(z-1)-z}{(z-1)^2}\right] = \frac{Tz}{(z-1)^2}$$

THE INVERSE Z-TRANSFORMS

The z-transforms serve the same role for discrete time control systems that the Laplace transforms serves for continuous time control systems. For the z-transform to be useful, we must be familiar with the methods of finding inverse z-transform. The notation for the inverse z-transform is Z^{-1}. the inverse z-transform of X(z)gives the corresponding time sequence x(k) at sampling instants. Thus, the inverse z-transform of X(z) yields a unique x(k), but does not yields a unique x(t). this means that the inverse z-transform yields a time sequence that specifies the values of x(t) only at discrete instants of time $t = 0, T, 2T, \ldots$ and say nothing about the values of x(t) at all other times. That is, many different time functions x(t) can have same x(kT).

Given the z-transform function X(z), the inverse z-transform of X(z) is denoted by $x(k) = Z^{-1}[X(z)]$.

In general, the inverse z-transform can be carried out with one of the following four methods.

(i) Z-transforms table

(ii) Partial fraction expression method

(iii) Power series or direct division method

(iv) Counter integration or inversion integral method.

(i) Z-Transform table

For simple functions, the inverse z-transform can be looked up from the z-transform table.

(i) Z-Transform table

For simple functions, the inverse z-transform can be looked up from the z-transform table.

Problem-18: Evaluate the inverse z-transform of $e^{1/z} + \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$

Solution:

$$Z^{-1}\left[e^{1/z} + \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right] = Z^{-1}(e^{1/z}) + Z^{-1}\left(\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right) = \frac{1}{k!} - \cos k\theta$$

(ii) Partial fraction expansion method

The partial fraction expansion method is parallel to the partial fraction expansion method used in the Laplace transform, with a minor modification. The z-transform X(z) can e expanded into partial fractions and using z-transform table corresponding inverse z-transform can be obtained. As indicated in z-transform table 'z' appears in the numerator of the z-transform and as such it is convenient to expand $\dfrac{X(z)}{z}$ into partial fractions instead of X(z).

Problem-19: Evaluate $Z^{-1}\left[\dfrac{3z^2+z}{(5z-1)(5z+2)}\right]$

Solution: Let $X(z) = \dfrac{3z^2+z}{(5z-1)(5z+2)}$

$$\frac{X(z)}{z} = \frac{3z+1}{(5z-1)(5z+2)} = \frac{A}{5z-1} + \frac{B}{5Z+2}$$

$$= \frac{8}{15}.\frac{1}{5z-1} + \frac{1}{15}.\frac{1}{5z+2}$$

$$= \frac{8}{75}\left(\frac{1}{z-1/5}\right) + \frac{1}{75}\frac{1}{(z+2/5)}$$

$$\Rightarrow X(z) = \frac{8z}{75(z-0.2)} + \frac{z}{75(z+0.4)}$$

$$\therefore Z^{-1}[X(z)] = Z^{-1}\left[\frac{8(z)}{75(z-0.2)} + \frac{z}{75(z+0.4)}\right] = \frac{8}{75}(0.2)^k + \frac{1}{75}(-0.4)^k$$

(iii) Power series or direct division method

In the direct division method, we obtain the inverse z-transform by expanding X(z) into an infinite power series in Z^{-1}. This method is useful when it is difficult to obtain the closed form expression for the inverse z-transform or it is desired to find only the first several terms of x(k). In general the z-transform equation can be given by

$$X(z)=\sum_{k=0}^{\infty}x(kT)z^{-k}=x(0)+x(2T)z^{-2}+........+x(kT)z^{-k}\rightarrow(23)$$

The given function X(z) can be expressed as a ratio of two polynomials in 'z' as follows

$$X(z)=\frac{a_0z^n+a_1z^{n-1}+a_2z^{n-2}+.........+a_n}{b_0z^m+b_1z^{m-1}+b_2z^{m-2}+.........+b_m}$$

If $n\leq m$, then dividing the numerator by denominator the above eqn.(2.24), can be written as

$$X(z)=x_0z^0+x_1z^{-1}+x_2z^{-2}+.......$$

By comparing the coefficients of power series of eqn.(2.25) and the z-transforms expression of eqn(2.23), it is concluded that the value of x(kT) at sampling instants i.e., x(0), x(T), x(kT)……,whereas the partial fraction expansion method gives an expression for x(k) in closed form.

Problem-21: Find x(k) for k=0,1,2,3,4 when x(z) is given by $X(z)=\frac{10z+5}{(z-1)(z-0.2)}$ by using direct division method

Solution: $X(z)=\frac{10z+5}{(z-1)(z-0.2)}=\frac{z(10+5/z)}{z^2(1-1/z)(1-0.2/z)}=\frac{10z^{-1}+5z^{-2}}{1-1.2z^{-2}+0.2^{-2}}$

$$1-1.2z^{-1}+0.2z^{-2}\;\Big)\;10z^{-1}+5z^{-2}\;\Big(\;10z^{-1}+17z^{-2}+18.4z^{-3}+18.68z^{-4}$$

$$\begin{array}{l}
10z^{-1}+5z^{-2}\\
\underline{10z^{-1}-12z^{-2}+2z^{-3}}\\
17z^{-2}-2z^{-3}\\
\underline{17z^{-2}-20.4z^{-3}+3.4z^{-4}}\\
18.42z^{-3}-3.4z^{-4}\\
\underline{18.42z^{-3}-22.08z^{-4}+3.666z^{-5}}\\
18.68z^{-4}-3.66z^{-5}\\
\underline{18.68z^{-4}-22.25z^{-5}+3.736z^{-6}}\\
18.73z^{-5}-3.736z^{-6}
\end{array}$$

$$\therefore X(z)=10z^{-1}+17z^{-2}+18.4z^{-3}+18.68z^{-4}+\frac{18.73z^{-5}-3.736z^{-6}}{1-1.2z^{-1}+0.2z^{-2}}$$

$\Rightarrow$ x(0)=0, x(1)=10, x(2)=17, x(3)=18.4, x(4)=18.68

Z-TRANSFORM METHOD FOR SOLVING DIFFERENCE EQUATIONS

Difference equations can be solved easily by the use of a digital computer provided the numerical values of all coefficients and parameters are given. However, closed form expressions for x(k) cannot be obtained from the computer solution, except for very special cases. The usefulness of the z- transform method is that it enables us to obtain the closed form expression x(k).

Consider the linear time invariant discrete time system characterized by the following linear difference equation.

$$x(k) + a_1 x(k-1) + \ldots\ldots\ldots\ldots + a_n x(k-n) = b_0 u(k) + b_1 u(k-1) + \ldots\ldots\ldots\ldots\ldots + b_n u(k-n)$$

Where u(k) and x(k) are the system's input and output respectively, at the k^{th} iteration. In describing such a difference equation in the z-plane, we take the z-transform of each term in the equation.

Problem-23: Solve the following difference equation by use of the z- transform method.

$$x(k+z) + zx(k+1) + zx(k) = 0,\ x(0) = 0,\ x(1) = 1$$

Solution: Taking the z- transform on both sides of the given difference equation, we obtain

$$z^2 X(z) - z^2 X(0) - zX(1) + 3zX(z) - 3zX(0) + 2X(z) = 0$$

$$z^2 X(z) - z + 3zX(z) + 2X(z) = 0$$

$$X(z)[z^2 + 3z + 2] = z$$

$$X(z) = \frac{z}{z^2 + 3z + 2} = \frac{z}{(z+1)(z+2)}$$

$$= z\left[\frac{1}{z+1} - \frac{1}{z+2}\right] = \frac{z}{z+1} - \frac{z}{z+2}$$

On taking inverse Z-transform on both side

$$x(k) = (-1)^k - (-2)^k$$

LIMITATIONS OF THE Z-TRANSFORM METHOD

The following considerations should be kept in mind while applying the Z-transform.

(i) Ideal sampler assumption

The derivation of the Z-transform of a continuous data function x(t) is based first on the sampling of the function by an ideal sampler. The result of this is that the Z-transform X(Z) represent the function x(t) only at the sampling instants.

(ii) Non uniqueness of the Inverse Z-transform

Given X(z), the inverse Z-transform of X(z) gives only a unique solution to x(kT). Strictly the solution to x(t) is unknown.

(iii) Accuracy of the Z-transform method

The accuracy of the Z-transform method depends on the magnitude of the sampling frequency ω_s or the sampling period T, relative to the highest frequency component contained in the function x(t).if the sampling period is too large or sampling frequency is too low, relative to the variation of x(t),the z-transform solution may be erroneous , since $x^*(t)$ would not be a good representative of x(t) at the sampling instants.

ADDITIONAL SOLVED PROBLEMS

ASP-1: Find out $Z^{-1}\left[\dfrac{10}{(z+1)(z+2)}\right]$

Solution: Let $X(z)=\left[\dfrac{10}{(z+1)(z+2)}\right]$

The simple poles are given by

$$z_1=-1, z_2=-2$$

$$\text{Re}\,sidue, K_1 = \underset{z\to-1}{Lt}\left[(z+1)\frac{10}{(z+1)(z+2)}\times z^{k-1}\right]$$

$$= \underset{z\to-1}{Lt}\left[\frac{10z^{k-1}}{(z+2)}\right]=\frac{10(-1)^{k-1}}{1}-10(-1)^k$$

$$\text{Re}\,sidue, K_2 = \underset{z\to-2}{Lt}\left[(z+2)\frac{10}{(z+1)(z+2)}\times z^{k-1}\right]$$

$$= \underset{z\to-2}{Lt}\left[\frac{10z^{k-1}}{(z+1)}\right]=\frac{-10(-2)^{k-1}}{1}-5(-2)^k$$

ASP-2: Given $X(s)=\dfrac{1}{s^2(s+1)}$, obtain the X(z) by using residues?

Solution:-

Residue Method:

- If X(s) has poles $s_1, s_2, ...s_n$ and $s = s_i$ is a simple pole, then the corresponding residue is given by $K_i = \underset{s \to s_i}{lt}(s - s_i)X(s)\dfrac{z}{z - e^{Ts}}$
- If X(s) has multiple pole S_j of order q ,then the corresponding residue is given by

$$K_j = \frac{1}{(q-1)!}\underset{s \to sj}{Lt}\frac{d^{q-1}}{ds^{q-1}}\left[(s - sj)^q X(s)\frac{Z}{Z - e^{Ts}}\right]$$

Given that $X(s) = \dfrac{1}{s^2(s+1)}$

The given system has simple pole at $s_1 = -1$ and multiple pole at $s_2 = 0$

$$\operatorname{Re}sidue, K_1 = \underset{s \to -1}{Lt}\left[(s+1)\frac{10}{(s+1)(s^2)} \times \frac{z}{z - e^{Ts}}\right]$$

$$= \underset{s \to -1}{Lt}\left[\frac{1}{s^2} \times \frac{z}{z - e^{Ts}}\right] = \frac{z}{z - e^{-T}}$$

$$\operatorname{Re}sidue, K_2 = \frac{1}{(2-1)!}\underset{s \to 0}{Lt}\frac{d}{ds}\left[(s-0)^2\frac{1}{s^2(s+1)}\frac{z}{z - e^{Ts}}\right]$$

$$= \underset{s \to 0}{Lt}\frac{d}{ds}\left[\frac{1}{s+1}\left(\frac{z}{z - e^{Ts}}\right)\right]$$

$$= \underset{s \to 0}{Lt}\, z\frac{d}{ds}\left[(s+1)^{-1}(z - e^{Ts})^{-1}\right]$$

$$= \underset{s \to 0}{lt}\left[-(s+1)^{-1}(z - e^{Ts})^{-2}(-e^{Ts}) + (z - e^{Ts})^{-1}(-1)(s+1)^{-2}\right]$$

$$= z\left[\frac{T}{(z-1)^2} - \frac{1}{z-1}\right]$$

$$\therefore X(z) = Z\left[\frac{1}{s^2(s+1)}\right] = sum\ of\ residues = \frac{z}{z - e^{-T}} + \frac{Tz}{(z-1)^2} - \frac{z}{z-1}$$

Assignment-Cum-Tutorial Questions

A. Questions testing the remembering / understanding level of students

I) Objective Questions

1. The Z-Transform X(z) of a discrete time signal x(n) is defined as:
   a) $\sum_{n=-\infty}^{\infty} x(n)z^{n}$
   b) $\sum_{n=-\infty}^{\infty} x(n)z^{-n}$
   c) $\sum_{n=0}^{\infty} x(n)z^{n}$
   d) None of the mentioned

2. The z-transform of a sequence x(n) which is given as $X(z)= \sum_{n=-\infty}^{\infty} x(n)z^{-n}$, is known as:
a) Uni-lateral Z-transform
b) Bi-lateral Z-transform
c) Tri-lateral Z-transform
d) None of the mentioned

3. Is the discrete time LTI system with impulse response $h(n)=a^n(n)$ ($|a| < 1$) BIBO stable?
a) True
b) False

4. Which of the following justifies the linearity property of z-transform?[$x(n)\leftrightarrow X(z)$].
a) $x(n)+y(n) \leftrightarrow X(z)Y(z)$
b) $x(n)+y(n) \leftrightarrow X(z)+Y(z)$
c) $x(n)y(n) \leftrightarrow X(z)+Y(z)$
d) $x(n)y(n) \leftrightarrow X(z)Y(z)$

5. What is the z-transform of the signal $x(n)=[3(2^n)-4(3^n)]u(n)$?
a) $3/(1-2z^{-1})-4/(1-3z^{-1})$
b) $3/(1+2z^{-1})-4/(1+3z^{-1})$
c) $3/(1-2z)-4/(1-3z)$
d) None of the mentioned

6. What is the z-transform of the signal $x(n)=\sin(j\omega_0 n)u(n)$?
a) $\frac{z^{-1}\sin\omega_0}{1+2z^{-1}\cos\omega_0+z^{-2}}$

b) $\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0-z^{-2}}$

c) $\frac{z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$

d) $\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$

ans:d

7. According to Time shifting property of z-transform, if X(z) is the z-transform of x(n) then what is the z-transform of x(n-k)?

a) $z^{k}X(z)$
b) $z^{-k}X(z)$
c) X(z-k)
d) X(z+k)

8. What is the z-transform of the signal defined as x(n)=u(n)-u(n-N)?

a) $\frac{1+z^{N}}{1+z^{-1}}$

b) $\frac{1-z^{N}}{1+z^{-1}}$

c) $\frac{1+z^{-N}}{1+z^{-1}}$

d) $\frac{1-z^{-N}}{1-z^{-1}}$

Ans:d

9. Causal systems are the systems in which

a. The output of the system depends on the present and the past inputs
b. The output of the system depends only on the present inputs
c. The output of the system depends only on the past inputs
d. The output of the system depends on the present input as well as the previous outputs

10. Time reversal of a discrete time signal refers to

a.y[n] = x[-n+k]
***b.** y[n] = x[-n]*
c. y[n] = x[-n-k]
d. y[n] = x[n-k]

II) Descriptive Questions

1.Differentiate continuous time and discrete time systems.

2.Explain the advantages and disadvantages of digital control systems.

3. With a suitable block diagram explain the general sampled data control system.

4. State and prove the following theorems of z-transforms i) Shifting theorem ii) Complex translation theorem iii) Partial differentiation theorem.

5. Explain the properties of z-transform.
6. What are the popular methods that are used to find the inverse z-transform? Explain briefly each of them.

7. Explain pulse transfer function matrix.

B. Question testing the ability of students in applying the concepts.
I) Objective Questions

1. If X(z) is the z-transform of the signal x(n) then what is the z-transform of $a^n x(n)$?
   a) X(az)
   b) $X(az^{-1})$
   c) $X(a^{-1}z)$
   d) None of the mentioned

2. What is the z-transform of the signal $x(n)=a^n(\sin\omega_0 n)u(n)$?
   a) $\frac{az^{-1}\sin\omega_0}{1+2az^{-1}\cos\omega_0+a^2z^{-2}}$

   b) $\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0-a^2z^{-2}}$

   c) $\frac{(az)^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$

   d) $\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$

   Ans:d

3. If X(z) is the z-transform of the signal x(n), then what is the z-transform of the signal x(-n)?
   a) X(-z)
   b) $X(z^{-1})$
   c) $X^{-1}(z)$
   d) None of the mentioned

4. What is the z-transform of the signal $x(n)=na^nu(n)$?

a) $\frac{(az)^{-1}}{(1-(az)^{-1})^2}$

b) $\frac{az^{-1}}{(1-(az)^{-1})^2}$

c) $\frac{az^{-1}}{(1-az^{-1})^2}$

d) $\frac{az^{-1}}{(1+az^{-1})^2}$

Ans:c

5. Which of the following method is used to find the inverse z-transform of a signal?
a) Counter integration
b) Expansion into a series of terms
c) Partial fraction expansion
d) All of the mentioned

6. What is the partial fraction expansion of the proper function $X(z)= 1/(1-1.5z^{-1}+0.5z^{-2})$?
a) 2z/(z-1)-z/(z+0.5)
b) 2z/(z-1)+z/(z-0.5)
c) 2z/(z-1)+z/(z+0.5)
d) 2z/(z-1)-z/(z-0.5)

7. What is the partial fraction expansion of $X(z)=1/((1+z^{-1})(1-z^{-1})^2)$?
a) $z/(4(z+1)) + 3z/(4(z-1)) + z/(2⟦(z+1)⟧^2)$
b) $z/(4(z+1)) + 3z/(4(z-1)) – z/(2⟦(z+1)⟧^2)$
c) $z/(4(z+1)) + 3z/(4(z-1)) + z/(2⟦(z-1)⟧^2)$
d) $z/(4(z+1)) + z/(4(z-1)) + z/(2⟦(z+1)⟧^2)$

8. The region of convergence of the z – transform of a unit step function is

(a) $|z| > 1$ (b) $|z| < 1$ (c) (Real part of z) > 0 (d)(Real part of z) < 0

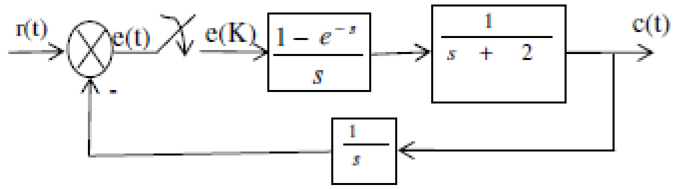
9. Time shifting of discrete time signal means

a. y[n] = x[n-k]
b. y[n] = x[-n-k]
c. y[n] = -x[n-k]
d. y[n] = x[n+k]

II) Descriptive Questions

1. Find the z-transform of $f(t)=e^{-at}\sin bt$.
2. Find the z-transform of $f(t)=t\sin\omega t$.
3. Find the inverse z-transform of $F(z)=\dfrac{5z}{z^2+4z+3}$.
4. Find the inverse z-transform of $F(z)=\dfrac{3z^2+2z+1}{z^2-3z+2}$.

5. For the sampled data control system shown in below figure, find the response to unit step input



6. For the sampled data system show in below figure, find the response to unit step input