STATE SPACE ANALYSIS

INTRODUCTION

The analysis and design of control system are carried out using transfer functions together with a variety of graphical techniques such as root locus plots and Nyquist plots based on the input-output relations of the system. They are applicable only to linear time invariant systems having a single input and single output (8180). Hence a new approach to control system analysis and design is evolved, which can be applied to the design of optimal and adaptive control system, which are mostly time varying and/or non-linear multiple inputs and multiple outputs(M1MO). This new approach is based on the concept of state, which includes the initial conditions in the design.

Advantages of state-space technique:

1) It is possible to analyse time-varying or time-invarying linear or non-linear, single or multiple input-output systems.
2) State equations are highly compatible for simulation on analog or digital computer.
3) It is possible to optimise the system useful for optimal design.
4) State space analysis gives us the information about the internal behaviour of the system,

   as well as the input and output behaviour.

CONCEPT OF STATE, STATE VARIABLES & STATE VECTOR

State: The state of a dynamic system is the smallest set of variables, called state variables such that the knowledge of these variables at $t = t_o$ together with the input for $t > t_o$. Completely determine the behaviour of the system for anytime $t > t_o$.

Note that in dealing with linear time invariant systems, we usually choose the reference time to be zero.

State Variables: The state variables of a dynamic system are the smallest set of variables which determine the state of the dynamic system. If at least n variables $x_1(k)$, $x_2(k)$,----------- $x_n(k)$ are needed to completely describe the behaviour of a dynamic system, then such n variables $x_1(k)$, $x_2(k)$,-----------$x_n(k)$ are called a set of state variables.

State Vector: If n state variables are needed to completely describe the behaviour of a given system then these n state variables can be considered to be the n components of a vector x(t). Such a vector is called a state vector.

State Space: The n-dimensional space whose coordinate axes consists of the x_1- axis, x_2-axis,

---- x_n-axis is called a state-space. Any state can be represented by a point in the state-space.

STATE SPACE EQUATIONS

In the state space analysis we are concerned with three types of variables that are involved in the modelling of dynamic systems: input variables, output variables and state variables.

For time varying (linear and non-linear) discrete time systems, the state equations may be written as

$$x(k+1) = f[x(k), u(k), k]$$

and the output equations as

$$y(k) = g[x(k), u(k), k]$$

For linear time varying discrete time systems, the state equation and output equation may be simplified to

$$\left.\begin{aligned} x(k+1) &= G(k)x(k) + H(k)u(k) \\ y(k) &= C(k)x(k) + D(k)u(k) \end{aligned}\right\}$$

Where,

x (k) = n-vector (state vector)

y (k) = m-vector (output vector)

u (k) = r-vector (state vector)

G (k) = n×n matrix (state vector)

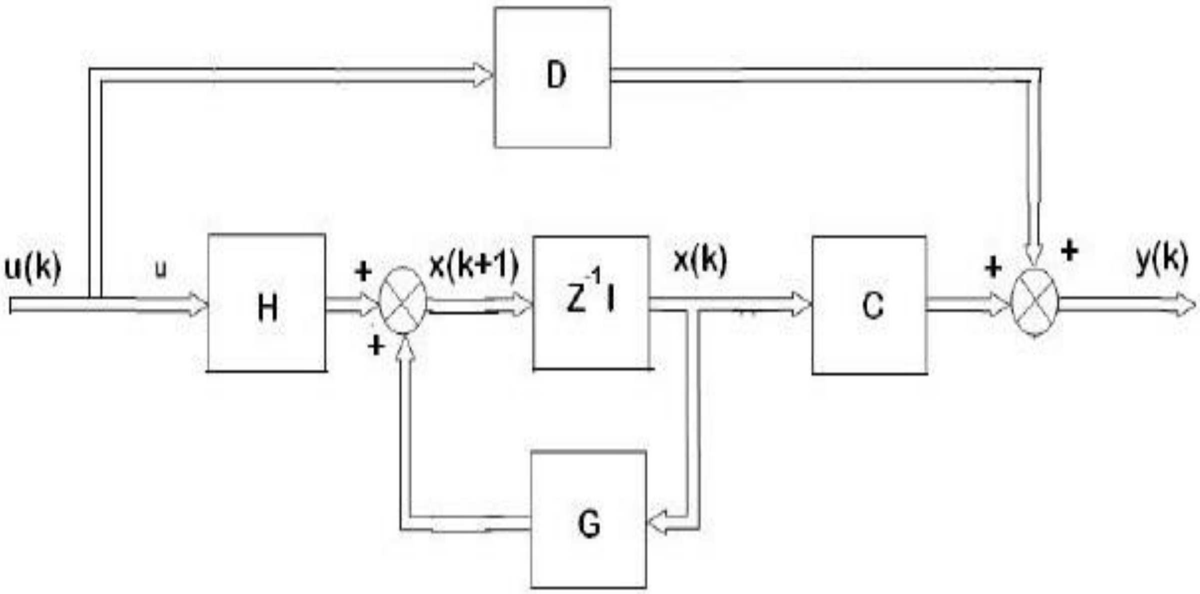
H (k) = n×r vector (input vector)

C (k) = m×n matrix (output matrix)

D (K) = m×r matrix (direct transmission)

For linear time invarying discrete time systems, the state equation and output equation may be simplified to

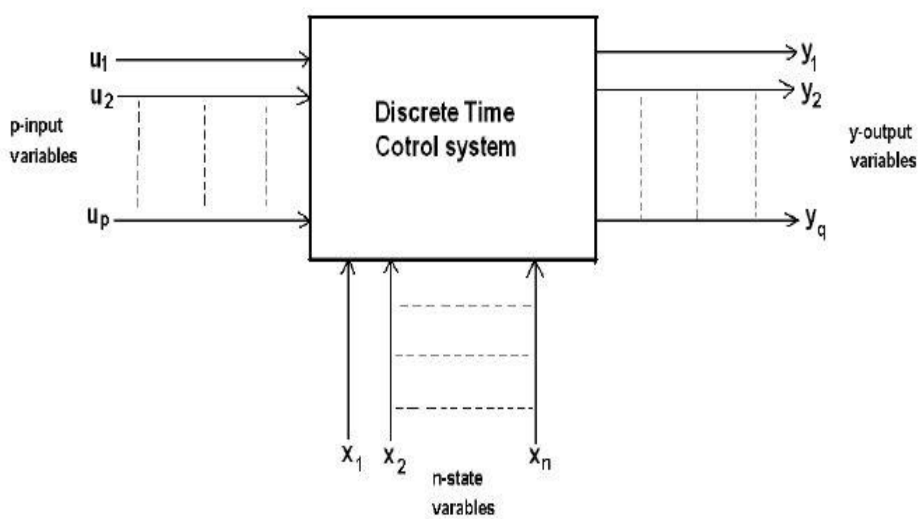
$$\left.\begin{aligned} x(k+1) &= Gx(k) + Hu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}\right\}$$



Block-diagram of a LTI discrete time system in state space

DISCRETE TIME STATE-SPACE EQUATINS

In state variable formulation, the state variables are generally represented by $x_1(k)$, $x_2(k)$, ------- $x_n(k)$; inputs by $u_1(k)$, $u_2(k)$, ------- $u_p(k)$; and the outputs by $y_1(k)$, $y_2(k)$, ------- $y_q(k)$, as shown in fig.(4.2).



Structure of general discrete time system

The input vector u(k), output vector y(k) and state vector x(k) are:

$$u(k)=\begin{bmatrix}u_1(k)\\u_2(k)\\\vdots\\u_p(k)\end{bmatrix},\quad y(k)=\begin{bmatrix}y_1(k)\\y_2(k)\\\vdots\\y_q(k)\end{bmatrix},\quad x(k)=\begin{bmatrix}x_1(k)\\x_2(k)\\\vdots\\x_n(k)\end{bmatrix}$$

The dynamics of an LTI system is described by the following set of equations

$$\left.\begin{array}{l}x_1(k+1)=g_{11}x_1(k)+g_{12}x_2(k)+\cdots+g_{1n}x_n(k)+h_{11}u_1(k)+h_{12}u_2(k)+\cdots+h_{1n}u_p(k)\\x_2(k+1)=g_{21}x_1(k)+g_{22}x_2(k)+\cdots+g_{2n}x_n(k)+h_{21}u_1(k)+h_{22}u_2(k)+\cdots+g_{1n}u_p(k)\\\vdots\qquad\qquad\qquad\qquad\vdots\qquad\qquad\qquad\qquad\vdots\\\vdots\qquad\qquad\qquad\qquad\vdots\qquad\qquad\qquad\qquad\vdots\\x_n(k+1)=g_{n1}x_1(k)+g_{n2}x_2(k)+\cdots+g_{nn}x_n(k)+h_{n1}u_1(k)+h_{n2}u_2(k)+\cdots+h_{nn}u_p(k)\end{array}\right\}$$

$$\left.\begin{array}{l}y_1(k)=c_{11}x_1(k)+c_{12}x_2(k)+\cdots+c_{1n}x_n(k)+d_{11}u_1(k)+d_{12}u_2(k)+\cdots+d_{1p}u_p(k)\\y_2(k)=c_{21}x_1(k)+c_{22}x_2(k)+\cdots+c_{2n}x_n(k)+d_{21}u_1(k)+d_{22}u_2(k)+\cdots+d_{1p}u_p(k)\\\vdots\qquad\qquad\qquad\qquad\vdots\qquad\qquad\qquad\qquad\vdots\\\vdots\qquad\qquad\qquad\qquad\vdots\qquad\qquad\qquad\qquad\vdots\\y_n(k)=c_{q1}x_1(k)+c_{q2}x_2(k)+\cdots+c_{qn}x_n(k)+d_{q1}u_1(k)+d_{q2}u_2(k)+\cdots+d_{qp}u_p(k)\end{array}\right\}$$

Where the coefficients g_{ij} h_{ij}, c_{ij} and d_{ij} are constants

In vector form above equations can be represented as

$$\left.\begin{array}{r}x(k+1)=Gx(k)+Hu(k)\\y(k)=Cx(k)+Du(k)\end{array}\right\}$$

Where

SELECTION OF STATE VARIABLES

The state variables of a given system are not unique. There are infinitely many choices for any given system. Following are the some of the guide lines to choose the state variables of a given system.

- If a physical system such as an electrical system, the number of state variables needed to represent the system must be equal to the number of energy storing elements present in the system.
- If a system is represented by a linear difference equation, then the number of state variables needed to represent the system must be equal to the order of the difference equation.
- If a system is represented by a pulse transfer function, then the number of state variables needed to represent the system must be equal to highest power of 'z' in the denominator of the pulse transfer function.

STATE-SPACE REPRESENTATIONS OF DISCRETE-TIME SYSTEM

Many techniques are available for obtaining state-space representation of discrete-time system. Consider the discrete –time system described by

$$y(k)+a_1y(k-1)+a_2y(k-2)+...a_ny(k-n)=b_0u(k)+b_1u(k-1)+b_2u(k-2)+\cdots+b_nu(k-n)$$

The above equation can be written in the form of pulse transfer function as

$$\frac{Y(z)}{U(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_nz^{-n}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}} \quad (or)$$

$$\frac{Y(z)}{U(z)}=\frac{b_0z^n+b_1z^{n-1}+b_2z^{n-2}+\cdots+b_n}{z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n} \qquad \text{--- (4.9)}$$

There are many ways to realize state-space representations for the discrete time system described by above equation.

a) Controllable canonical form/direct programming method

b) Observable canonical form/nested programming method

c) Diagonal canonical form/ Jordan canonical form /partial fraction expansion method

(a) Controllable canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ . \\ . \\ . \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & . & . & . & a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ . \\ . \\ . \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & . & . & . & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ . \\ . \\ . \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

(b) Observable canonical form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ . \\ . \\ . \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & . & . & 0 & 0 & -a_n \\ 1 & 0 & . & . & 0. & 0 & -a_{n-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 & -a_2 \\ 1 & 0 & . & . & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ . \\ . \\ . \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ . \\ . \\ . \\ . \\ b_1 - a_1 b_0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & . & . & . & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ . \\ . \\ . \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

(c) Diagonal canonical form

If the poles of the pulse transfer function given by above equations are all distinct, then the state-space representation may be put in the diagonal form as follows:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & p_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & p_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} c_1 & c_2 & \cdot & \cdot & \cdot & c_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

Jordan canonical form: If the pulse transfer function given by above equations involves a multiple pole of order m at $z = p_1$ and all other poles are distinct, then the state equation and output equation may be given as follows:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_m(k+1) \\ x_{m+1}(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & p_1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_{m+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & p_m \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_m(k) \\ x_{m+1}(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

Problem-1: For the given pulse transfer function, $\frac{Y(z)}{U(z)} = \frac{z+1}{z^2+1.3z+0.4}$, obtain the state-space representation in

i) Controllable canonical form/direct programming method/phase variable form

ii) Observable canonical form/nested programming method

iii) Diagonal canonical form/partial fraction expansion method/Jordan canonical form

Solution:

i) Controllable canonical form/direct programming method

$$\frac{Y(z)}{U(z)} = \frac{z+1}{z^2+1.3z+0.4} = \frac{z(z^{-1}+1)}{z^2(1+1.3z^{-1}+0.4z^{-2})}$$

$$\frac{Y(z)}{U(z)} = \frac{z^{-1}+z^{-2}}{1+1.3z^{-1}+0.4z^{-2}}$$

$$\frac{Y(z)}{z^{-1}+z^{-2}} = \frac{U(z)}{1+1.3z^{-1}+0.4z^{-2}} = Q(z)$$

$$U(z) = (1+1.3z^{-1}+0.4z^{-2})Q(z)$$

$$\Rightarrow Q(z) = -1.3z^{-1}Q(z) - 0.4z^{-2}Q(z) + U(z) \qquad \text{--- (1)}$$

$$Y(z) = (z^{-1}+z^{-2})Q(z)$$

$$Y(z) = z^{-1}Q(z) + z^{-2}Q(z) \qquad \text{--- (2)}$$

Let

$$z^{-2}Q(z) = X_1(z)$$

$$z^{-1}Q(z) = X_2(z) \Rightarrow Q(z) = zX_2(z)$$

$$X_1(z) = z^{-1}X_2(z) \Rightarrow zX_1(z) = X_2(z)$$

$$\Rightarrow x_1(k+1) = x_2(k) \qquad \text{--- (3)}$$

From eqn.(1)

$$zX_2(z) = -1.3z^{-1}zX_2(z) - 0.4X_1(z) + U(z)$$

$$x_2(k+1) = -0.4x_1(k) - 1.3x_2(k) + u(k) \qquad \text{--- (4)}$$

From eqn.(2)

$$Y(z) = X_1(z) + X_2(z)$$

$$y(k) = x_1(k) + x_2(k) \qquad \text{--- (5)}$$

$$\therefore x_1(k+1) = x_2(k)$$

$$x_2(k+1) = -0.4x_1(k) - 1.3x_2(k) + u(k)$$

$$y(k) = x_1(k) + x_2(k)$$

(ii) Observable canonical form

$$\frac{Y(z)}{U(z)} = \frac{z^{-1} + z^{-2}}{1 + 1.3z^{-1} + 0.4z^{-2}}$$

$$(1 + 1.3z^{-1} + 0.4z^{-2})Y(z) = (z^{-1} + z^{-2})U(z)$$

$$Y(z) = -1.3z^{-1}Y(z) - 0.4z^{-2}Y(z) + z^{-1}U(z) + z^{-2}U(z)$$

$$= z^{-1}[U(z) - 1.3Y(z)] + z^{-2}[U(z) - 0.4Y(z)]$$

$$Y(z) = z^{-1}\{[U(z) - 1.3Y(z)] + z^{-1}[U(z) - 0.4Y(z)]\} \text{--- (1)}$$

Let $Y(z) = X_2(z)$

$$\Rightarrow X_2(z) = z^{-1}\{[U(z) - 1.3Y(z)] + z^{-1}[U(z) - 0.4Y(z)]\}$$

$$\left.\begin{aligned} X_2(z) &= z^{-1}[U(z) - 1.3Y(z) + X_1(z)] \\ X_1(z) &= z^{-1}[U(z) - 0.4Y(z)] \end{aligned}\right\} \qquad \text{--- (2)}$$

From eqn.(2)

$$zX_2(z) = U(z) - 1.3X_2(z) + X_1(z)$$

$$\Rightarrow x_2(k+1) = u(k) - 1.3x_2(k) + x_1(k) \qquad \text{--- (3)}$$

$$zX_1(z) = U(z) - 0.4X_2(z)$$

$$\Rightarrow x_2(k+1) = u(k) - 1.3x_2(k) + x_1(k) \qquad \text{--- (3)}$$

$$zX_1(z) = U(z) - 0.4X_2(z)$$

$$x_1(k+1) = u(k) - 0.4x_2(k) \qquad \text{--- (4)}$$

From eqn.(1)

$$y(k) = x_2(k) \qquad \text{--- (5)}$$

$$\therefore x_1(k+1) = u(k) - 0.4x_2(k)$$

$$x_2(k+1) = u(k) - 1.3x_2(k) + x_1(k)$$

$$y(k) = x_2(k)$$

In state-space form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

(iii) Diagonal canonical form/partial fraction expansion method/Jordan canonical form

$$\frac{Y(z)}{U(z)} = \frac{z+1}{z^2+1.3z+0.4} = \frac{z+1}{(z+0.5)(z+0.8)} = \frac{5/3}{(z+0.5)} + \frac{-2/3}{(z+0.8)}$$

$$Y(z) = \frac{5}{3}\frac{U(z)}{(z+0.5)} - \frac{2}{3}\frac{U(z)}{(z+0.8)}$$

$$\Rightarrow y(k) = \frac{5}{3}x_1(k) - \frac{2}{3}x_2(k) \qquad \text{--- (1)}$$

Where

$$X_1(z) = \frac{U(z)}{(z+0.5)}$$

$$zX_1(z) = u(z) - 0.5X_1(z)$$

$$\Rightarrow x_1(k+1) = -.5x_1(k) + u(k) \qquad \text{--- (2)}$$

$$X_2(z) = \frac{U(z)}{(z+0.8)}$$

$$zX_2(z) = U(z) - 0.8X_1(z)$$

$$x_2(k+1) = -0.8x_2(k) + u(k) \qquad \text{--- (3)}$$

$$\therefore x_1(k+1) = -.5x_1(k) + u(k)$$

$$x_2(k+1) = -0.8x_2(k) + u(k)$$

$$y(k) = \frac{5}{3}x_1(k) - \frac{2}{3}x_2(k)$$

In state-space form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \frac{5}{3} & \frac{-2}{3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

SOLUTION OF LINEAR TIME INVARIANT DISCRETE TIME STATE EQUATION

Recursion Procedure

In general, discrete time equation is easier to solve than differential equations because the former can be solved easily by means of recursion procedure. The recursion procedure is quite simple and convenient for digital computations.

Consider the following state equation and output equation

$$x(k+1) = Gx(k) + Hu(k)$$

$$y(k) = Cx(k) + Du(k)$$

Put k=0

$$x(1) = Gx(0) + Hu(0)$$

For k=1

$$\begin{aligned} x(2) &= Gx(1) + Hu(1) \\ &= G[Gx(0) + Hu(0)] + Hu(1) \\ &= G^2x(0) + GHu(0) + Hu(1) \end{aligned}$$

For k=2

$$\begin{aligned} x(3) &= Gx(2) + Hu(2) \\ &= G[G^2x(0) + GHu(0) + Hu(1)] + Hu(2) \\ &= G^3x(0) + G^2Hu(0) + GHu(1) + Hu(2) \\ &\vdots \qquad\qquad\qquad \vdots \end{aligned}$$

$$x(k) = G^k x(0) + \sum_{j=0}^{k-1} G^{k-j-1} Hu(j), k = 1,2,3....$$

Let $\Phi(k) = G^k$

$$x(k) = \Phi(k)x(0) + \sum_{j=0}^{k-1} \Phi(k-j-1)Hu(j)$$

Clearly, x(k) consist of two parts, one representing the contribution of the initial state x(0) and the other the contribution of the input u(j).

The output y(k) is given by

$$y(k) = CG^k x(0) + C\sum_{j=0}^{k-1} G^{k-j-1} Hu(j) + Du(k)$$

Here $\Phi(k)$ =G^k= State Transition Matrix (STM) or Fundamental Matrix

Z-Transform Method

Consider the discrete time system described by

$$x(k+1) = Gx(k) + Hu(k)$$

Taking Z-transforms on both sides

z X(z)- z x(0)=G X(z)+HU(z)

[zI-G] X(z)= z x(0)+HU(z)

Pre-multiplying both sides with $[zI-G]^{-1}$

On comparing above equations we will obtain

Φ(k)=G^k=Z^{-1} [$(zI-G)^{-1}$z]=S.T.M

$$\sum_{j=0}^{k-1} G^{k-j-1}Hu(j) = Z^{-1}[(zI-G)^{-1}HU(z)],\ k=1,2,3\text{----------}$$

Notice that the solution by the Z-transform method involves the process of inverting the matrix (zI-G), which may be accomplished by analytical means or by use of a computer. The solution also requires the inverse Z-transforms of $(zI-G)^{-1}z$ and $(zI-G)^{-1}HU(z)$

Problem-2: Obtain the state transition matrix of the following discrete time system

$$x(k+1) = Gx(k) + Hu(k)$$

$$y(k) = Cx(k)$$

Where

$$G = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Then obtain the state x(k) and the output y(k) when the input u(k)=1 for k=0,1,2,-----.

Assume that the initial state is given by $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solution: Given that

$$G = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Initial state, $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$zI - G = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} = \begin{bmatrix} z & -1 \\ 0.16 & z+1 \end{bmatrix}$$

$$(zI - G)^{-1} = \frac{1}{z(z+1)+0.16}\begin{bmatrix} z+1 & 1 \\ -0.16 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{z+1}{(z+0.8)(z+0.2)} & -\frac{1}{(z+0.8)(z+0.2)} \\ \frac{-0.16}{(z+0.8)(z+0.2)} & \frac{2}{(z+0.8)(z+0.2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3(z+0.2)} - \frac{1}{3(z+0.8)} & \frac{5}{3(z+0.2)} - \frac{5}{3(z+0.8)} \\ \frac{-0.8}{3(z+0.2)} + \frac{0.8}{3(z+0.8)} & \frac{-1}{3(z+0.2)} + \frac{4}{3(z+0.8)} \end{bmatrix}$$

$$\Phi(k) = Z^{-1}\left[(zI - G)^{-1}z\right] = Z^{-1}\begin{bmatrix} \frac{4z}{3(z+0.2)} - \frac{z}{3(z+0.8)} & \frac{5z}{3(z+0.2)} - \frac{5z}{3(z+0.8)} \\ \frac{-0.8z}{3(z+0.2)} + \frac{0.8z}{3(z+0.8)} & \frac{-z}{3(z+0.2)} + \frac{4z}{3(z+0.8)} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{4}{3}(-0.2)^k - \frac{1}{3}(-0.8)^k & \frac{5}{3}(-0.2)^k - \frac{5}{3}(-0.8)^k \\ \frac{-0.8}{3}(-0.2)^k + \frac{0.8}{3}(-0.8)^k & \frac{-1}{3}(-0.2)^k + \frac{4}{3}(-0.8)^k \end{bmatrix}$$

$$Z[x(k)] = X(z) = (zI-G)^{-1}zx(0) + (zI-G)^{-1}HU(z)$$

$$= (zI-G)^{-1}[zx(0) + HU(z)]$$

Since u(k) =1 $\Rightarrow U(z) = \frac{z}{z-1}$

$$zx(0) + HU(z) = \begin{bmatrix} z \\ -z \end{bmatrix} + \begin{bmatrix} \frac{z}{z-1} \\ \frac{z}{z-1} \end{bmatrix} = \begin{bmatrix} z + \frac{z}{z-1} \\ -z + \frac{z}{z-1} \end{bmatrix} = \begin{bmatrix} \frac{z^2}{z-1} \\ \frac{-z^2+2z}{z-1} \end{bmatrix}$$

$$X(z) = (zI-G)^{-1}[zx(0) + HU(z)]$$

$$= \begin{bmatrix} \frac{z+1}{(z+0.2)(z+0.8)} & \frac{1}{(z+0.2)(z+0.8)} \\ \frac{-0.16}{(z+0.2)(z+0.8)} & \frac{z}{(z+0.2)(z+0.8)} \end{bmatrix} \begin{bmatrix} \frac{z^2}{z-1} \\ \frac{-z^2+2z}{z-1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(z^2+z)z}{(z+0.2)(z+0.8)(z-1)} \\ \frac{(-z^2+1.84z)z}{(z+0.2)(z+0.8)(z-1)} \end{bmatrix} = \begin{bmatrix} \frac{-17z}{6(z+0.2)} + \frac{22z}{9(z+0.8)} + \frac{25z}{18(z-1)} \\ \frac{34z}{6(z+0.2)} - \frac{17.6z}{9(z+0.8)} + \frac{7z}{18(z-1)} \end{bmatrix}$$

$$x(k) = Z^{-1}[X(z)] = \begin{bmatrix} \frac{-17}{6}(-0.2)^k + \frac{22}{9}(-0.8)^k + \frac{25}{18}(1)^k \\ \frac{34}{6}(-0.2)^k - \frac{17.6}{9}(-0.8)^k + \frac{7}{18}(1)^k \end{bmatrix}$$

The output is given by

$y(k) = Cx(k)$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{-17}{6}(-0.2)^k + \frac{22}{9}(-0.8)^k + \frac{25}{18}(1)^k \\ \frac{34}{6}(-0.2)^k - \frac{17.6}{9}(-0.8)^k + \frac{7}{18}(1)^k \end{bmatrix}$$

$$= \frac{-17}{6}(-0.2)^k + \frac{22}{9}(-0.8)^k + \frac{25}{18}(1)^k$$

Clearly $\Phi(k) = G^k = Z^{-1}[(zI-G)^{-1}z]$

Therefore $\Phi(k)$ is called as State Transition Matrix or also called as Fundamental Matrix.

Properties of State Transition Matrix:

(i) $\Phi(0) = I$

$\Phi(k) = G^k$, $\Phi(0) = G^0 = I$

(ii) $\Phi(k_1 + k_2) = \Phi(k_1)\ \Phi(k_2)$

$\Phi(k_1 + k_2) = G^{k_1+k_2} = G^{k_1} \times G^{k_2} = \Phi(k_1)\Phi(k_2)$

(iii) $\Phi(-k) = \Phi^{-1}(k)$

$\Phi(-k) = G^{-k} = (G^k)^{-1} = \Phi^{-1}(k)$

(iv) $\Phi(nk) = \Phi^n(k)$

$\Phi(nk) = G^{nk} = (G^k)^n = \Phi^n(k)$

Method of computing STM

The state transition matrix is given by

$$\Phi(k) = G^k = Z^{-1}\left[(zI - G)^{-1} z\right]$$

The following two methods are used to find STM

(i) The Caley-Hamilton Theorem

(ii) The z-transform method

(i) The Caley-Hamilton Theorem

The theorem states that every square matrix must satisfy its own characteristic equation. For example, the characteristics equation of the matrix G(n×n) is written as

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + --------- + a_1 z + a_0 = 0$$

Then

$$G^n + a_{n-1}G^{n-1} + a_{n-2}G^{n-2} + --------- + a_1 G + a_0 I = 0$$

$$\Rightarrow STM = G^k = -[a_{k-1}G^{k-1} + a_{k-2}G^{k-2} + --------- + a_1 G + a_0 I]$$

(ii) The z-transform method

In z-transform method, the STM is given by

$$\Phi(k) = Z^{-1}\left[(zI - G)^{-1} z\right]$$

Method for computing $(zI\text{-}G)^{-1}$:

The method presented here is based on the expansion of adjoint of $(zI - G)$.The inverse of $(zI - G)$ can be written in terms of adjoint of $(zI - G)$), as follows:

$$(zI - G)^{-1} = \frac{adj(zI - G)}{\det(zI - G)}$$

Note that the determinant |zI-G| may be written as follows

$$|zI - G| = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + ---------- + a_1 z + a_0$$

The adjoint of (ZI-G) can be given by

$$adj(zI - G) = Iz^{n-1} + H_1 z^{n-2} + H_2 z^{n-3} + \ldots + H_{n-1}$$

Where

$$\left.\begin{aligned}
H_1 &= G + a_1 I \\
H_2 &= GH_1 + a_2 I \\
\vdots \quad & \quad \vdots \\
H_{n-1} &= GH_{n-2} + a_{n-1} I \\
H_n &= GH_{n-1} + a_n I
\end{aligned}\right\}$$

Note that $a_1, a_2, \ldots a_n$ are the coefficients appearing in the determinant

The a_i's can also be given by use of the trace, as follows:

$$\left.\begin{aligned}
a_1 &= -tr[G] \\
a_2 &= -\frac{1}{2} tr[GH_1] \\
a_3 &= -\frac{1}{3} tr[GH_2] \\
\vdots \quad & \quad \vdots \\
a_n &= -\frac{1}{n} tr[GH_{n-1}]
\end{aligned}\right\}$$

since $a_1, H_1, a_2, H_2, \ldots, a_{n-1}, H_{n-1}$ can be easily computed sequentially.

Problem-3: The state equation of a digital control system is defined by

$x(k+1) = Gx(k) + Hu(k)$

Where $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$. Determine State transition matrix (STM)?

Solution:

$$(zI - G)^{-1} = \frac{Adj(zI - G)}{|zI - G|}$$

$$\Delta = |zI - G| = \left| \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \right| = \left| \begin{bmatrix} z & -1 & 0 \\ 0 & z & -1 \\ 6 & 11 & z+6 \end{bmatrix} \right|$$

$$= z(z^2 + 6z + 11) + 6$$

$$= z^3 + 6z^2 + 11z + 6$$

$$= (z+1)(z+2)(z+3)$$

$$= z^3 + a_1 z^2 + a_2 z + a_3$$

$$\Rightarrow a_1 = 6, a_2 = 11, a_3 = 6$$

$$Adj(zI - G) = Iz^2 + H_1 z + H_2$$

$$H_1 = G + a_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + 6 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ -6 & -11 & 0 \end{bmatrix}$$

$$H_2 = GH_1 + a_2 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ -6 & -11 & 0 \end{bmatrix} + 11 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 & 1 \\ -6 & 0 & 0 \\ 0 & -6 & 0 \end{bmatrix}$$

$$Adj(zI-G) = Iz^2 + H_1 z + H_2 = \begin{bmatrix} z^2 & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & z^2 \end{bmatrix} + \begin{bmatrix} 6z & z & 0 \\ 0 & 6z & z \\ -6z & -11z & 0 \end{bmatrix} + \begin{bmatrix} 11 & 6 & 1 \\ -6 & 0 & 0 \\ 0 & -6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} z^2+6z+11 & z+6 & 1 \\ -6 & z^2+6z & z \\ -6z & -11z-6 & z^2 \end{bmatrix}$$

$$(zI-G)^{-1} = \frac{Adj(zI-G)}{|zI-G|} = \begin{bmatrix} \frac{z^2+6z+11}{(z+1)(z+2)(z+3)} & \frac{z+6}{(z+1)(z+2)(z+3)} & \frac{1}{(z+1)(z+2)(z+3)} \\ \frac{-6}{(z+1)(z+2)(z+3)} & \frac{z^2+6z}{(z+1)(z+2)(z+3)} & \frac{z}{(z+1)(z+2)(z+3)} \\ \frac{-6z}{(z+1)(z+2)(z+3)} & \frac{-11z-6}{(z+1)(z+2)(z+3)} & \frac{z^2}{(z+1)(z+2)(z+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{z+1} - \frac{3}{z+2} + \frac{1}{z+3} & \frac{2.5}{z+1} - \frac{4}{z+2} + \frac{1.5}{z+3} & \frac{0.5}{z+1} - \frac{1}{z+2} + \frac{0.5}{z+3} \\ \frac{-3}{z+1} + \frac{6}{z+2} + \frac{3}{z+3} & \frac{-2.5}{z+1} + \frac{8}{z+2} + \frac{3}{z+3} & \frac{-0.5}{z+1} + \frac{2}{z+2} + \frac{1.5}{z+3} \\ \frac{3}{z+1} - \frac{12}{z+2} - \frac{9}{z+3} & \frac{2.5}{z+1} - \frac{16}{z+2} + \frac{13.5}{z+3} & \frac{0.5}{z+1} - \frac{4}{z+2} + \frac{4.5}{z+3} \end{bmatrix}$$

State transition matrix is given by

$$\phi(kT) = Z^{-1}\left[(zI-G)^{-1} z\right]$$

$$= Z^{-1}\begin{bmatrix} \frac{3z}{z+1} - \frac{3z}{z+2} + \frac{z}{z+3} & \frac{2.5z}{z+1} - \frac{4z}{z+2} + \frac{1.5z}{z+3} & \frac{0.5z}{z+1} - \frac{z}{z+2} + \frac{0.5z}{z+3} \\ \frac{-3z}{z+1} + \frac{6z}{z+2} + \frac{3z}{z+3} & \frac{-2.5z}{z+1} + \frac{8z}{z+2} + \frac{3z}{z+3} & \frac{-0.5z}{z+1} + \frac{2z}{z+2} + \frac{1.5z}{z+3} \\ \frac{3z}{z+1} - \frac{12z}{z+2} - \frac{9z}{z+3} & \frac{2.5z}{z+1} - \frac{16z}{z+2} + \frac{13.5z}{z+3} & \frac{0.5z}{z+1} - \frac{4z}{z+2} + \frac{4.5z}{z+3} \end{bmatrix}$$

$$= \begin{bmatrix} 3(-1)^k - 3(-2)^k + (-3)^k & 2.5(-1)^k - 4(-2)^k + 1.5(-3)^k & 0.5(-1)^k - (-2)^k + 0.5(-3)^k \\ -3(-1)^k + 6(-2)^k + 3(-3)^k & -2.5(-1)^k + 8(-2)^k + 3(-3)^k & -0.5(-1)^k + 2(-2)^k + 1.5(-3)^k \\ 3(-1)^k + 12(-2)^k - 9(-3)^k & 2.5(-1)^k - 16(-2)^k + 13.5(-3)^k & 0.5(-1)^k - 4(-2)^k + 4.5(-3)^k \end{bmatrix}$$

DISCRETIZATION OF CONTINUOUS TIME STATE-SPACE EQUATIONS

Review of Solution of Continuous time state-space equations

The state equation of the continuous time system is

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\dot{x}(t) - Ax(t) = Bu(t)$$

Premultiplying both sides with e^{-At}

$$e^{-At}[\dot{x}(t) - Ax(t)] = \frac{d}{dt}\left[e^{-At}u(t)\right] = e^{-At}Bu(t)$$

Integrating the above equation between the limits 0 and t

$$\int_0^t \frac{d}{dt}[e^{-At}x(t)]dt = \int_0^t e^{-At}Bu(t)$$

$$e^{-At}x(t) + x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$x(t) = e^{-At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Note that the solution of the state equation starting with the initial state $x(t_0)$ is

$$x(t) = e^{-A(t-t_0)}x(t_0) + \int_{t_0}^t e^{-A(t-\tau)}Bu(\tau)d\tau$$

Discretization of continuous time state space equations

Consider the continuous time state equation and output equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y = Cx + Du$$

In the following analysis to clarify the presentation, we use the notation kT and (k+1)T instead of k and k+1. The discrete time representation of eqn.(4.73) will be in the form

$$x[(k+1)T] = G(T)x(kT) + H(T)u(kT)$$

Note that the matrices G and H depend on the sampling period 'T'. Once the sampling period T is fixed, G and H are constant matrices.

To determine G(T) and H(T), We assume that the input u(t) is sampled and fed to zero-order hold so that all the components of u(t) are constant over the interval between any two consecutive sampling instants , or

$$u(t) = u(kT) \text{ for } kT \le t < kT + T$$

Since

$$x[(k+1)T] = e^{A(k+1)T}x(0) + e^{A(k+1)T}\int_0^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau$$

And

$$x(kT) = e^{AkT}x(0) + e^{AkT}\int_0^{kT} Bu(\tau)d\tau$$

Eq(3) - $e^{AT} \times$ eq(4)

$$\Rightarrow x[(k+1)T] - e^{AT}x(kT) = e^{A(k+1)T}\int_0^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau - e^{A(k+1)T}\int_0^{kT} e^{-A\tau}Bu(\tau)d\tau$$

$$x(k+1)T = e^{AT}x(kT) + e^{A(k+1)T}\int_{kT}^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau$$

Since from eqn.(7), $u(t) = u(kT)$ for $kT \le t < kT + T$, we may substitute $u(\tau) = u(kT) =$ const. in this last equation . Hence we may write

$$x[(k+1)T] = e^{AT}\, x[kT] + e^{AT} \int_0^T e^{-AT} Bu(kT)dt$$

$$= e^{AT} x(kT) + \int_0^T e^{A\lambda} Bu(kT)d\lambda$$

Where $\lambda = \text{T-t}$

If we define

$$G(T) = e^{AT}$$

$$H(T) = \left(\int_0^T e^{A\lambda} d\lambda \right) B$$

$$x[(k+1)t] = G(T)x(kT) + H(T)u(kT)$$

$$y(kT) = Cx(kT) + Du(kT)$$

Where matrices C and D are constant matrices and do not depend on the sampling period T.

Problem- 4: Consider the continuous time system given by $G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+a}$. Obtain the continuous time state space representation of the system. Then discretized the state equation and output equation and obtain the discrete time state space representation of the system.

Solution: Given that

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+a}$$

$$(s+a)Y(s) = U(s)$$

Take inverse L.Ts on both sides

$$\frac{dy}{dt} + ay = u(t)$$

Let us choose y=x, state variable

$$\dot{x} = -ax + u$$

$$y = x$$

$$\therefore\ G(T) = e^{-aT}$$

$$H(T) = \left(\int_0^T e^{A\lambda}\, d\lambda\right)B = \int_0^T e^{-a\lambda} d\lambda = \left[\frac{e^{-a\lambda}}{-a}\right]_0^T = \frac{1-e^{-aT}}{a}$$

Hence, the discretized version of the system equation is

$$x[(k+1)T] = G(T)x(kT) + H(T)u(kT)$$

$$x(k+1) = e^{-aT}x(k) + \frac{1-e^{-aT}}{a}u(k)$$

$$y(k) = x(k)$$

CONCEPT OF CONTROLLABILITY AND OBSERVABILITY

INTRODUCTION

Controllability is concerned with the problem of whether it is possible to steer a system from a given initial state to an arbitrary state. A system is said to be controllable if it is possible by means of an unbounded control vector to transfer the system from any initial state to any other state in a finite number of sampling periods.

Observability is concerned with the problem of determining the state of a dynamic system from the observation of the output and control vectors in a finite number of sampling periods. A system is said to be observable if, with the system in state x(0), it is possible to determine this state from the observation of the output and control vectors over a finite number of sampling periods.

The concept of controllability and observability were introduced by R.E.Kalman. They play an important role in the optimal control of multivariable systems.

CONTROLLABILITY

A control system is said to be completely state controllable if it is possible to transfer the system from any arbitrary initial state to any desired state in a finite time period. That is, a control system is controllable if every state variable can be controlled in a finite time period by some unconstrained control signal. If any state variable is independent of the control signal, then it is impossible to control this state variable and therefore the system is uncontrollable.

Complete State Controllability

Consider the discrete time control system defined by

$$\left.\begin{aligned} x[(k+1)T] &= Gx(kT) + Hu(kT) \\ y(kT) &= Cx(kT) + Du(kT) \end{aligned}\right\}$$

The system described by eqn.(5.1) is said to be completely state controllable or simply state controllable , if for any initial time KT=0, there exists a set of a unconstrained control signal u(KT), which transfers the state x(KT) from any initial state x(0) to final state x(N) for some finite time N.

Complete Output Controllability

The system described by eqn.(5.1) is said to be completely output controllable, if for any initial time kT=0, there exists a set of unconstrained control signal u(kT), k=0,1,2,.....N-1, such that any final output y(N) can be reached from arbitrary initial states in a finite time N.

THEOREMS ON CONTROLLABILITY

1 Complete State Controllability

$$\left.\begin{aligned} x[(k+1)T] &= Gx(kT) + Hu(kT) \\ y(kT) &= Cx(kT) + Du(kT) \end{aligned}\right\}$$

The system described by eqn.(5.2) is completely state controllable if and only if, the matrix $Q_c = [G^0H : GH : G^2H : \ldots\ldots\ldots\ldots : G^{n-1}H]$ is of rank n. The matrix Q_c is called State Controllability Matrix.

Proof: The solution of the state equation is

$$x(nT) = G^n x(0) + \sum_{j=0}^{n-1} G^{n-j-1} Hu(jT)$$

$$= G^n x(0) + G^{n-1}H\,u(0) + G^{n-2}H\,u(T) + \ldots\ldots\ldots\ldots\ldots\ldots + H\,u[(n-1)T]$$

$$X(nT) - G^n x(0) = [H\ GH\ \ldots..\ G^{n-1}H]\begin{bmatrix} u(n-1)T \\ u(n-2)T \\ . \\ . \\ u(0) \end{bmatrix}$$

If the rank of matrix $Q_c = [H\ GH\ \ldots\ldots\ldots\ldots\ G^{n-1}H]$ is n, then the matrix Q_c is called the Controllability matrix. Thus, if the rank of the Controllability matrix is n, then for an arbitrary state $x(nT) = x_f$, there exists a sequence of unbounded control signals u(0), u(T),…………….., u[(n-1)T] that satisfies eqn.(5.3). Hence, the condition that the rank of the controllability matrix be 'n' gives a sufficient condition for Complete State Controllability.

Complete output controllability

$$\left.\begin{aligned} x[(k+1)T] &= Gx(kT) + Hu(kT) \\ y(kT) &= Cx(kT) + Du(kT) \end{aligned}\right\}$$

The system described by is completely output controllable if and only if the following matrix is of rank p, where p is the dimension of the output vector y(kT).

$$Q_{CO} = [D : CH : CGH : \ldots\ldots. : CG^{n-1}H]$$

The matrix Q_{CO} is referred to as the output controllability matrix.

Proof:

The solution of the above state equation is

$$x(nT) = G^n x(0) + \sum_{j=0}^{n-1} G^{n-1-j} Hu(jT)$$

$$y(nT) = Cx(nT) + Du(nT)$$

$$= C\,[G^n x(0) + \sum_{j=0}^{n-1} G^{n-1-j} Hu(jT)] + D\,u(nT)$$

$$y(nT) - CG^n x(0) = \{CG^{n-1}H\,u(0) + CG^{n-2}\,H\,u(2T) + \ldots\ldots\ldots\ldots CG^0H\,u[(n-1)T]\} + Du(nT)$$

$$y(nT) - CG^n x(0) = [D : CG^0H : CGH : \ldots\ldots. : CG^{n-1}H]\begin{bmatrix} u(nT) \\ u[(n-1)T] \\ . \\ . \\ u(0) \end{bmatrix}$$

Thus, a necessary and sufficient condition for the system defined by completely output controllable is that the $m\times(n+1)r$ matrix

$$[D : CG^0H : CGH : \ldots\ldots. : CG^{n-1}H]$$

Be of rank 'm'.

OBSERVABILITY

The system is said to be completely observable if every initial state x(0) can be determined from the observation of y(kT) over a finite number of sampling periods. The system, therefore, is completely observable if every transition of the state eventually affects every element of the output vector.

Problem-1: Check whether the system represented by

$$x(k+1) = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \quad \text{is controllable or not?}$$

Solution: From the given state space representation

$$G = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$GH = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$Q_C = \begin{bmatrix} H & GH \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$$

Rank of Q_C= r =1

Here we have, n=2

Hence r≠n and the system is uncontrollable.

Problem-2: A linear dynamic time invariant system is represented by

$$x(k+1) = Gx(k) + Hu(k)$$

Where $$G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}, H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Find if the system is completely controllable or not?

Solution: For the system, n=3

$$GH = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -3 & 0 \end{bmatrix}$$

$$G^2H = G\,(GH) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 7 & 0 \end{bmatrix}$$

$$Q_C = [H \;\; G\,H \;\; G^2H]$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 1 & 0 & -3 & 0 & 7 & 0 \end{bmatrix}$$

Consider the determinant, $\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 1 \neq 0$

Here the rank of the matrix, $Q_C = 3 = n$

Thus the given system is completely controllable.

THEOREMS ON OBSERVABILITY

2 Complete Observability

Consider the unforced discrete time system is described by

$$\left.\begin{aligned} x[(k+1)T] &= Gx(kT) \\ y(kT) &= Cx(kT) \end{aligned}\right\}$$

is completely observable if and only if the following matrix is of rank 'n'.

$$Q_o = [C^T : G^TC^T : \ldots\ldots : (G^T)^{n-1}C^T]$$

Proof:

The solution of the state equation is

$$x(kT) = G^K x(0)$$

$$\therefore y(kT) = CG^k x(0)$$

Complete observability means that, given y(0), y(T), y(2T)……, it is possible to determine $x_1(0)$, $x_2(0)$, ……, $x_n(0)$. To determine n unknowns, we need only n values of y(kT). Hence, we may use the first n values of y(kT) or y(0), y(T),……, y[(n-1)T] for determination of $x_1(0)$, $x_2(0)$, ……., $x_n(0)$.

For a completely observable system, given

$$\begin{aligned} y(0) &= C\,x(0) \\ y(T) &= C\,Gx(0) \\ &\vdots \\ &\vdots \\ y[(n-1)T] &= CG^{n-1}\,x(0) \end{aligned}$$

We must able to determine $x_1(0)$, $x_2(0)$, ……., $x_n(0)$.

$$\begin{bmatrix} y(0) \\ y(T) \\ . \\ . \\ y[(n-1)T] \end{bmatrix} = \begin{bmatrix} C \\ CG \\ . \\ . \\ CG^{n-1} \end{bmatrix} x(0)$$

The rank of the matrix $Q_O = \begin{bmatrix} C \\ CG \\ CG^2 \\ . \\ . \\ CG^{n-1} \end{bmatrix}$ (or) $Q_O = [C^T : G^TC^T : \ldots\ldots : (G^T)^{n-1}C^T]$ is n then it is observable. Here Q_O is called the Observability matrix.

Problem-3: Check whether the system represented by

$$x\,(k+1) = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} x\,(k) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} u\,(k)$$

$$y\,(k) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x\,(k)$$

Is observable or not?

Solution: From the given state space representation

$$G = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix}\ C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$G^T = \begin{bmatrix} -3 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\quad C^T = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$G^T C^T = \begin{bmatrix} -3 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(G^T)^2 C^T = G^T\,(G^T C^T) = \begin{bmatrix} -3 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -4 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 11 \\ 0 & -4 \\ 1 & -1 \end{bmatrix}$$

$$Q_O = [C^T\ \ G^T C^T\ \ (G^T)^2\,C^T]$$

$$= \begin{bmatrix} 0 & 1 & 0 & -4 & 0 & 11 \\ 0 & 1 & 0 & 1 & 0 & -4 \\ 1 & 0 & 1 & 2 & 1 & -1 \end{bmatrix}$$

We are able to choose a determinant from Q_O where value is not zero, which implies that, the rank of Q_O is 3. Here the system is completely observable.

RELATIONSHIP BETWEEN CONTROLLABILITY, OBSERVABLILITY AND TRANSFER FUNCTIONS

Classical analysis and design of control system uses the concept transfer functions for system modeling. One advantage of using transfer function is that state controllability and observability are directly related to the minimum order of the transfer function. The following theorem gives the relationship between controllability and observability and pole-zero cancellation of a transfer function.

Theorem (1)-Controllability, observability and transfer functions

If the input-output transfer functions of a linear time-invariant digital system has pole-zero cancellation, the system will be either not state controllable, unobservable, or both, depending on how the state variables are defined. If the input-output transfer function does not have pole-zero cancellation, then the system can always be represented by dynamic equations as a completely controllable and observable system.

Proof: Consider an nth order differential equation as given below.

$$\left.\begin{aligned} x(k+1) &= Gx(k) + Hu(k) \\ y(k) &= Cx(k) \end{aligned}\right\}$$

Let the matrix G be diagonalized by n×n non-singular matrix P, such that $x(k) = P\hat{x}(k)$

$$\left.\begin{aligned} \therefore P\hat{x}(k+1) &= GP\hat{x}(k) + H\,u(k) \\ y(k) &= CP\hat{x}(k) \end{aligned}\right\}$$

$$\left.\begin{aligned} \hat{x}(k+1) &= P^{-1}GP\hat{x}(k) + P^{-1}H\,u(k) \\ y(k) &= CP\hat{x}(k) \end{aligned}\right\}$$

$$\left.\begin{aligned} \Rightarrow \hat{x}(k+1) &= \hat{G}\hat{x}(k) + \hat{H}\,u(k) \\ y(k) &= \hat{C}\hat{x}(k) \end{aligned}\right\}$$

Where $\hat{G} = P^{-1}GP$, $\hat{H} = P^{-1}H$, $\hat{C} = CP$

Since $\hat{G}$ is a diagonal matrix, the i^{th} (i=1, 2………n) equation

$$\hat{x}_i(k+1) = z_i \hat{x}_i(k) + \gamma_i \; u(k)$$

Where z_i is the i^{th} eigen value of G and γ_i is the i^{th} element of $\hat{G}$

Taking the Z-transforms by assuring zero initial conditions

$$z \hat{X}_i(z) = z_i \hat{X}_i(z) + \gamma_i \; U(z)$$

$$\Rightarrow \hat{X}_i(z) = \frac{\gamma_i}{z - z_i} U(z)$$

The Z-transform , gives

$$Y(z) = CP \hat{X}(z)$$

Let $\hat{C}$ = CP = [f_1, f_2 -------f_n]

Where f_i = [$c_1 + c_2 z_i + \text{-------} + c_n z_i^{n-1}$]

$$Y(z) = CP \frac{\gamma_i}{z - z_i} U(z) = [f_1 \; f_2 \; .....f_n] \begin{bmatrix} \dfrac{\gamma_1}{z - z_1} \\ \dfrac{\gamma_2}{z - z_2} \\ \vdots \\ \dfrac{\gamma_n}{z - z_n} \end{bmatrix} U(z)$$

Or in the transfer function form

$$\frac{Y(z)}{U(z)} = \sum_{i=1}^{n} \frac{f_i \gamma_i}{z - z_i}$$

If the transfer function has a cancellation of pole and zero, the corresponding coefficient on the right side of the would be zero. Assuming that the pole at z = z_i is cancelled by a zero, then

$$f_j \gamma_j = 0$$

$$f_j = 0 \text{ or } \gamma_j = 0 \text{ or both.}$$

Since γ_j is the j th element of matrix $\hat{G}$, $\gamma_j = 0$ would mean that the system is uncontrollable.

On the on the other hand, if $f_j = 0$ where f_j is the j^{th} element of $\hat{C}$, the system would be unobservable.

Problem-5: y(k+2) + 2y(k+1) + y(k) = u(k+1) + u(k). Test the controllability and observability?

Solution: Given that

y(k+2) + 2y(k+1) + y(k) = u(k+1) + u(k)

Taking the Z-transforms on both the sides

z^2 Y(z) + 2z Y(z) + Y(z) = z U(z) + U(z)

Y(z) (z^2 + 2z + 1) = (z+1) U(z)

$$\Rightarrow \frac{Y(z)}{U(z)} = \frac{z+1}{z^2+2z+1} = \frac{z(1+z^{-1})}{z^2(1+2z^{-1}+z^{-2})} = \frac{z^{-1}+z^{-2}}{1+2z^{-1}+z^{-2}}$$

Controllable canonical form:

$$\frac{Y(z)}{z^{-1}+z^{-2}} = \frac{U(z)}{1+2z^{-1}+z^{-2}} = Q(z) \; let$$

$$U(z) = Q(z) [1 + 2z^{-1} + z^{-2}]$$

$$Q(z) = -2z^{-1} Q(z) - z^{-2} Q(z) + U(z)$$

Let

$$z^{-2} Q(Z) = X_1(z)$$

$$z^{-1} Q(Z) = X_2(z) \Rightarrow Q(z) = z X_2(z)$$

$$X_1(z) = z^{-1} X_2(z)$$

$$z X_1(z) = X_2(z) \Rightarrow x_1(k+1) = x_2(k)$$

$$C(z) = (z^{-1} + z^{-2}) Q(z)$$

$$C(z) = z^{-1} Q(z) + z^{-2} Q(z) = X_1(z) + X_2(z)$$

$$\Rightarrow y(k) = x_1(k) + x_2(k)$$

$$Q(z) = -2z^{-1} Q(z) - z^{-2} Q(z) + U(z)$$

$$zX_2(z) = -2X_2(z) - X_1(z) + U(z)$$

$$\Rightarrow x_2(k+1) = -x_1(k) - 2x_2(k) + u(k)$$

$$\therefore x_1(k+1) = x_2(k)$$

$$x_2(k+1) = -x_1(k) - 2x_2(k) + u(k)$$

$$y(k) = x_1(k) + x_2(k)$$

In state space form we can write as follows

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

From the above state-space representation

$$G = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$GH = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$Q_c = [H \quad GH] = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$|Q_c| = \begin{vmatrix} 0 & 1 \\ 1 & -3 \end{vmatrix} = 0 - 1 = -1 \neq 0$$

$\Rightarrow$ Rank = 2

$\therefore$ The system is completely state controllable.

$$G^T = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}, C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$G^T C^T = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$Q_O = [C^T \colon G^T C^T] = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$|Q_O| = 0$ and hence rank, $r = 1 \neq n$

$\therefore$ The given system unobservable.

Observable canonical form:

$$\frac{Y(z)}{U(z)} = \frac{z^{-1} + z^{-2}}{1 + 2z^{-1} + z^{-2}}$$

$$(1 + 2z^{-1} + z^{-2})Y(z) = (z^{-1} + z^{-2})U(z)$$

$$Y(z) = -2z^{-1}Y(z) - z^{-2}Y(z) + z^{-1}U(z) + z^{-2}U(z)$$

$$= z^{-1}[U(z) - 2Y(z)] + z^{-2}[U(z) - Y(z)]$$

$$Y(z) = z^{-1}\{[U(z) - 2Y(z)] + z^{-1}[U(z) - Y(z)]\} \qquad \text{--- (1)}$$

Let $Y(z) = X_2(z)$

$$\Rightarrow X_2(z) = z^{-1}\{[U(z) - 2Y(z)] + z^{-1}[U(z) - Y(z)]\}$$

$$\left.\begin{aligned} X_2(z) &= z^{-1}[U(z) - 2Y(z) + X_1(z)] \\ X_1(z) &= z^{-1}[U(z) - Y(z)] \end{aligned}\right\} \qquad \text{--- (2)}$$

From eqn.(2)

$$zX_2(z) = U(z) - 2X_2(z) + X_1(z)$$

$$\Rightarrow x_2(k+1) = u(k) - 2x_2(k) + x_1(k) \qquad \text{--- (3)}$$

$$zX_1(z) = U(z) - X_2(z)$$

$$x_1(k+1) = u(k) - x_2(k) \qquad \text{--- (4)}$$

From eqn.(1)

$$y(k) = x_2(k) \qquad \text{--- (5)}$$

$$\begin{aligned} \therefore x_1(k+1) &= -x_2(k) + u(k) \\ x_2(k+1) &= x_1(k) - 2x_2(k) + u(k) \\ y(k) &= x_2(k) \end{aligned}$$

In state-space form, the above equations can be represented as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

From the above state-space representation

$$G = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$GH = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$Q_C = \begin{bmatrix} H & GH \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$|Q_c| = \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 1 - 1 = 0$$

$\Rightarrow$ Rank = $1 \neq$ n

$\therefore$ The system is not state controllable.

$$G^T = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$G^T C^T = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$Q_O = [C^T : G^T C^T] = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$|Q_O|$ = -1 and hence rank, r=2= n

$\therefore$ The given system completely observable.

In the above problem, it is clear that for one form of state space representation the system is completely state controllable and unobservable and another form state space representation the system is uncontrollable and completely state observable. This is because there is a pole-zero cancellation in the given system transfer function.